

A NONCONTAMINATING ITERATIVE
PROCESS FOR DETERMINING THE
ZEROS OF A POLYNOMIAL

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CHAPTER I

INTRODUCTION

History and Statement of the Problem

In [3], a method was derived for the estimation of the zeros of a polynomial with real coefficients, $g(x) = \alpha x^{n+1} + f(x)$, from the known zeros of $f(x)$, assuming that α is a small real number. In [11] J. E. Maxfield et al., established an iterative process which converges to the zeros of $g(x)$ and can be used to give a better estimate of these zeros than the previous method. The present paper now aims to present an extension of [11]:

Let (1) $t(x)$ be a polynomial with known zeros $r_1, \dots, r_n$.

From $t(x) = \sum_{i=1}^{n+1} a_i x^{i-1}$ with $a_n = 1$ and the zeros, r_i , of $t(x)$ we wish to establish an iterative procedure which converges to the zeros, $r_1^*, r_2^*, \dots, r_{n+1}^*$, of

$$(2) \quad g(x) = \alpha_{n+2} x^{n+1} + f(x) \quad \text{where}$$

$$(3) \quad f(x) = \sum_{i=1}^{n+1} (a_i + \alpha_i) x^{i-1} \quad a_{n+1} = 1, \text{ and}$$

$\alpha_1, \alpha_2, \dots, \alpha_{n+2}$ are of small magnitude.

In this paper, the coefficients of $f(x)$, $g(x)$ and $t(x)$, as well as the increments α_j 's, need not be restricted to real numbers.

Moreover, as a process for determining the zeros, $r_1, \dots, r_n$, of $t(x)$, Lin's Penultimate Remaindering Iteration is studied, and it

shall be shown that a translation of the variable x can control its convergence.

Application of the Paper

In a problem for determining the zeros of $f(x)$ with real coefficients, most of the established methods that can be applied, such as Lin's Penultimate Remaindering, are contaminated factorization schemes in the sense that the degree of accuracy of one factor affects that of the succeeding factors. For instance, in Penultimate Remaindering, the depression of the polynomial after a factor is obtained causes error due to the rounding in the division process itself, as well as due to the inaccuracy, perhaps, of the first factor obtained. Thus most of these methods give only the estimate r_1 to the actual zeros r_1^* . If $t(x)$ is the polynomial having the r_1 's for its true zeros, then each coefficient of $t(x)$ differs from the corresponding coefficient of $f(x)$ by some small number α_1 . Using these α_1 's and r_1 's as input, the iterative procedure we derive here may be applied as a corrective measure to obtain the actual zeros r_1^* 's, of $g(x)|_{\alpha_{n+2}=0}$ which is precisely $f(x)$.

Aside from being merely corrective, the said iteration can also be used as a method by itself of determining the zeros of a polynomial, independent of any other method. In this case we take for the r_1 's rough estimates of the zeros being sought. For, even though the iteration shall be proved to converge only for the increments α_1 's small enough in absolute value, like other iterations, experience shows that surprisingly, even with very poor estimates of the zeros for a start, the iteration frequently converges.

Furthermore, suppose the zeros of $g(x) = \alpha_{n+2} x^{n+1} + f(x)$, where α_{n+2} is quite small in magnitude, are to be determined. We may truncate it temporarily by neglecting the leading term to obtain the polynomial $f(x)$ of lower degree. If we can factor this polynomial and with these zeros as r_i 's, we may then employ our iterative scheme to obtain the zeros of $g(x)$. The α_j 's are set to 0 except α_{n+2} in case the zeros of $f(x)$ are obtained to arbitrary precision.

Also, there are occasions when the zeros of $g(x) = \alpha_{n+2} x^{n+1} + f(x)$, α_{n+2} small in magnitude, are to be determined wherein somehow, the zeros of $f(x)$ have been previously known, solved exactly or estimated. For instance, in servomechanisms, guided missiles, and electronic circuits, it is usually expedient to simplify the differential system by neglecting higher-ordered derivatives; then frequently, after studying the truncated system, it is necessary to reintroduce the higher-ordered derivatives one at a time and study the augmented system. Here we can take $f(x)$ to be the characteristic function of the truncated system, and $g(x)$ that of the augmented system, that is, the differential equation $f(x)$ with a leading term adjoined corresponding to the next higher-ordered derivative previously neglected. Using the iterative procedure we derive here, we can get the zeros of $g(x)$ from the known zeros of $f(x)$.

CHAPTER II

CONTROL OF TRANSLATION ON PENULTIMATE REMAINDERING

Before deriving the iterative formulas for the zeros of

$g(x) = \alpha_{n+2} x^{n+1} + \sum_1^{n+1} (a_1 + \alpha_1) x^{i-1}$, let us first study a method for determining the zeros r_1 's of $\sum_1^{n+1} a_1 x^{i-1}$, which can be used as input to the iteration.

One of the most convenient methods for determining the zeros of a polynomial is Lin's Penultimate Remaindering [1], which is an iterative scheme of producing one factor of a polynomial at a time. Like most other methods though, it is a contaminating procedure as has been explained in the introduction. With the zeros obtained from this as input, the iteration which shall be derived in the succeeding chapters can serve as corrective means and give a more accurate set of zeros.

Let us now analyze the convergence of Penultimate Remaindering, specifically whether or not it can be controlled by translation of variable. We shall restrict our attention to an n th-degree polynomial $f(x)$ with real coefficients, which for the purpose of convenience stands for either the original polynomial whose zeros are to be determined or for one of its depressed forms, and the factor being sought be either linear or quadratic.

Our problem at this point can be explicitly stated as follows:

Does there exist a real number h for which Lin's Penultimate Remaindering shall converge for $f(x+h)$?

Let $f(x) = \sum_1^{n+1} a_i x^{i-1} = \prod_1^n (x - \gamma_i)$ and the divisor $d(x) =$

$(x - \beta_1)(x - \beta_2)$ where $\beta_k = \gamma_k$ for $k = 1, 2$. From [1], the eigenvalues of the matrix of errors in the iteration are

$$\rho_k = -c_1^{-1}(\beta_k^{n-2} + c_{n-2} \beta_k^{n-3} + \dots c_2 \beta_k), \quad k = 1, 2, \text{ where}$$

$$G(x) = f(x)/d(x) = x^{n-2} + c_{n-2} x^{n-3} + \dots c_2 x + c_1.$$

Similarly, if the divisor is linear, say $d(x) = (x - \beta_1)$ where

$$\beta_1 = \alpha_1, \quad \rho_1 = -c_1^{-1}(\beta_1^{n-1} + c_{n-1} \beta_1^{n-2} + \dots c_2 \beta_1) \text{ where}$$

$$G(x) = f(x)/d(x) = \sum_1^n c_i x^{i-1}, \quad c_n = 1.$$

Of course, these eigenvalues were derived on the assumption that at each stage of iteration, the deviations of the coefficients of the trial divisor from those of the true divisor are small enough to allow terms involving said deviations of order higher than one to be neglected.

To obtain the corresponding expressions for the eigenvalues $\rho_k(h)$ of $f(x+h)$, we use the fact that the zeros of $f(x+h)$ are $\gamma_j - h$, $j = 1, \dots, n$ and that ρ_k may be more conveniently written as

$$\rho_k = \frac{G(\beta_k) - c_1}{-c_1} = 1 - \frac{G(\beta_k)}{c_1} = 1 - \frac{\prod_1^n (\beta_k - \gamma_j)}{(-1)^{n-2} \prod_1^n \gamma_j} = 1 - \frac{\prod_1^n (\gamma_j - \beta_k)}{\prod_1^n \gamma_j}$$

for $d(x)$ quadratic, and $k = 1, 2$.

Similarly for $d(x)$ linear, $\rho_1 = 1 - \frac{\frac{n}{2} (\gamma_j - \beta_1)}{\frac{n}{2} \gamma_j}$.

So $\rho_k(h) = 1 - \frac{\frac{n}{3} (\gamma_j - h - \beta_k + h)}{\frac{n}{3} (\gamma_j - h)} = 1 - \frac{\frac{n}{3} (\gamma_j - \beta_k)}{\frac{n}{3} (\gamma_j - h)}$ for $d(x)$ quadratic;

Similarly, for $d(x)$ linear, $\rho_1(h) = -\frac{\frac{n}{2} (\gamma_j - \beta_1)}{\frac{n}{2} (\gamma_j - h)} + 1$.

Our objective then is to find a real value h for which $|\rho_k(h)| < 1$ for $k = 1, 2$ or for $k = 1$, according as $d(x)$ is quadratic or linear.

Case 1. There exists β_k equal to some γ_j , $j = 1, \dots$ or n

This implies that for any h , except possibly $h = \beta_k$ if real $\rho_k(h) = 1$. But h cannot be β_k , for if it were, $f(x+h)$ would have 0 for a zero and the Penultimate Remaindering cannot be applied in that situation. Hence there is no value of h for which the iteration would converge in this case.

To avoid such an exceptional case, however, we can first obtain all the multiple zeros of $f(x)$ and leave a depressed polynomial with simple zeros. One way to do this is to apply the Euclidean Algorithm for determining the greatest common divisor of $f(x)$ and its first derivative. To illustrate, if

$$f(x) = \prod_1^r (x - e_i)^{p_i} \quad p_i \geq 1,$$

$$f'(x) = \prod_1^r (x - e_i)^{p_i - 1} H(x) \text{ where } H \text{ is a polynomial of degree } \geq 0,$$

then $g(x) = (f, f') = \prod_1^r (x - e_1)^{p_1 - 1}$ whose zeros are the multiple zeros of $f(x)$. Here (h, k) stands for g.c.d. of h and k . If g is readily factorable, we can determine the multiple zeros of $f(x)$, including their respective multiplicities. Otherwise, we may as well consider $f(x)/g(x)$ as our original polynomial instead of $f(x)$, for the zeros of the former are simple and precisely the same as the zeros of the latter. The multiplicities of the zeros of $f(x)$ obtained by the iteration on $f(x)/g(x)$ can be determined later.

Case 2. All the zeros of $f(x)$ are simple

Let $\prod_3^n (\gamma_j - \beta_k) = A_k + B_k i$, $k = 1, 2$, where for each k A_k and B_k are not 0 simultaneously. Let the zeros of $f(x)$ be $\beta_1, \beta_2, \gamma_{2t-1} = c_t + d_t i$, and $\gamma_{2t} = c_t - d_t i$, where $t = 2, \dots, s$; $\gamma_j = c_j$ for $j = 2s+1, \dots, n$. Here s is an integer within the closed interval $[1, n/2]$ and the c 's and d 's are real numbers such that $d > 0$. A slight modification will be necessary in case $d(x)$ is linear, namely, that the complex zeros be γ_{2t} and γ_{2t+1} , t starting from 1, $\dots, s$; and the real zeros aside from $\gamma_1 = \beta_1$ be γ_j , $j = 2s+2, \dots, n$, where $s \in [0, \frac{n}{2}]$.

Three lemmas are worthy of note at this point:

Lemma 1. $\prod_3^n (\gamma_j - h)$ (or its corresponding expression when $d(x)$ is linear) is always real and nonzero. If all the zeros of $f(x)$ are complex, the said expression is positive.

Proof.

$$\begin{aligned} \prod_3^n (\gamma_j - h) &= \prod_t [(c_t - h) + d_t i][(c_t - h) - d_t i] \prod_j (c_j - h) \\ &= \prod_t [(c_t - h)^2 + d_t^2] \prod_j (c_j - h), \end{aligned}$$

which is clearly real, and nonzero provided $h \neq$ any zero of $f(x)$.

In the above the empty product = 1. If all zeros are complex,

$$\prod_3^n (\gamma_j - h) = \prod_t [(c_t - h)^2 + d_t^2], \text{ which is positive.}$$

Lemma 2. When β_k 's are real, $\prod_3^n (\gamma_j - \beta_k)$ (or its corresponding expression when $d(x)$ is linear) are real and nonzero. For two successive real β_k 's, they are of the same sign, positive when the β_k 's are the two least real zeros.

Proof. The first statement follows from the proof of (1), replacing h by β_k . To show the other, let us for the purpose of convenience let π_k stand for $\prod_3^n (\gamma_j - \beta_k)$. Note that the sign of π_k depends only on the second product $\prod_j (c_j - \beta_k)$. For β_k to be the two least real zeros, each factor in that product is positive, so π_k is positive, for each k . When the β_k 's are not necessarily the least but still successive, there would be as many negative factors in π_1 as in π_2 , hence they would be of the same sign.

Lemma 3. If the β_k 's are complex conjugates, π_k 's are complex conjugates, too.

Proof. To see this, let $\beta_1 = c_1 + d_1 i = \overline{\beta}_2$. Expand π_1 as

$$\prod_2^s [(c_t - c_1) + (d_t - d_1)i] [(c_t - c_1) + (-d_t - d_1)i] \prod_{2s+1}^n [(c_j - c_1) - d_1 i] =$$

$$\prod_2^s [(c_t - c_1)^2 + d_t^2 - d_1^2 - 2d_1(c_t - c_1)i] \prod_{2s+1}^n [(c_j - c_1) - d_1 i].$$

On the other hand,

$$\pi_2 = \prod_2^s [(c_t - c_1) + (d_t + d_1)i] [(c_t - c_1) + (-d_t + d_1)i] \prod_{2s+1}^n [(c_j - c_1) + d_1 i] =$$

$$\prod_2^s [(c_t - c_1)^2 + d_t^2 - d_1^2 + 2d_1(c_t - c_1)i] \prod_{2s+1}^n [(c_j - c_1) + d_1 i]. \text{ Hence } \pi_1 = \overline{\pi}_2.$$

Let us now go back to our main problem, that of showing the existence of a real value h for which $|\rho_k(h)| < 1$, $k = 1, 2$.

Theorem 1. Let $f(x)$ be an n th-degree polynomial with real coefficients and simple zeros and having at least one real zero. Then there exists a positive number h for which Penultimate Remaindering, applied to $f(x+h)$, will converge to a linear or quadratic factor according as n is odd or even.

Proof. If n is odd, we know there is at least one real zero.

Let β_1 be the smallest such. Then,

$$\rho_1(h) = 1 - \frac{\frac{n}{2} (\gamma_j - \beta_1)}{\frac{n}{2} (\gamma_j - h)}$$

where, as pointed out in Lemma 2,

$$A = \frac{n}{2} (\gamma_j - \beta_1) > 0.$$

Since $\frac{n}{2} (\gamma_j - h)$ must be positive to have $|\rho_1(h)| < 1$, let it be $C > 0$.

$$|\rho_1(h)| = |1 - \frac{A}{C}| < 1 \iff |C-A| < C \iff -C < C-A < C \iff C > A/2.$$

C has to be big enough so that the equation $\frac{n}{2} (\gamma_j - h) = C$ would have a real solution in h . For practical purposes, we can estimate h provided we know an estimate β' of β_1 , for since $A = G(\beta_1)$, where $G = f(x)/d(x)$, A can be estimated by $g(\beta')$, where $g(x) = f(x)/(x-\beta') \approx G(x)$, from which C can be estimated, and consequently, h can be estimated by

$\frac{n-1}{\sqrt{C}}$ provided C is large enough.

Henceforth, let us assume n even. The rest of the real zeros, if any, may be obtained in a similar fashion as above, either singly or in successive pairs starting from the smallest. If in pairs,

$$\rho_k(h) = 1 - \frac{\frac{n}{3} (\gamma_j - \beta_k)}{\frac{n}{3} (\gamma_j - h)} = 1 - \frac{A_k}{C}$$

where $A_1, A_2 > 0$, from Lemma 2. C must necessarily be positive, too, to have $|\rho_k(h)| < 1$. Moreover,

$$|\rho_k(h)| < 1 \iff |C - A_k| < C \iff -C < C - A_k < C \iff C > A_k/2, \quad k = 1, 2.$$

$$\iff C > \max_k \{A_k/2\}.$$

So by choosing C large enough, we can estimate h by $\frac{n-2}{\sqrt{C}}$.

Thus we have shown that a real zero or a pair of real zeros can always be made "stable" by a proper choice of h . By a stable zero is meant one wherein the iteration can converge to its corresponding factor. Therefore, all the real zeros of $f(x)$ can be obtained using Penultimate Remaindering, with translation on x , if necessary.

Theorem 2. Let $f(x)$ be an n th-degree polynomial with real coefficients and with all its zeros complex and simple. Then there exists a positive number h for which Penultimate Remaindering, applied to $f(x+h)$, will converge to a factor corresponding to a pair of complex zeros.

Proof.

$$\rho_k(h) = 1 - \frac{\pi_k}{C} \quad \text{where } \pi_1 = A + Bi \text{ and } \pi_2 = A - Bi,$$

A, B real and $C > 0$ using Lemmas 1 and 3. Since $\rho_1(h)$ and $\rho_2(h)$ are complex conjugates, $|\rho_1(h)| = |\rho_2(h)|$, so that we need to consider only one of them.

$$\rho_1(h) = 1 - \frac{A+Bi}{C} = (1 - A/C) - (B/C)i.$$

From this, it is clear that A must necessarily be positive in order to have $|\rho_1(h)| < 1$, for if $A < 0$,

$$|\rho_1(h)|^2 = (1-A/C)^2 + (B/C)^2 > 1 \text{ since } A/C < 0, \text{ from } C > 0.$$

Using Lemma 4 below, we know there exist two zeros of $f(x)$ for which $A > 0$; hence we choose these two zeros as β_1 and β_2 .

$$\begin{aligned} |\rho_k(h)| < 1 &\Leftrightarrow (1-A/C)^2 + (B/C)^2 < 1 \Leftrightarrow 1 - \frac{2A}{C} + \frac{A^2}{C^2} + \frac{B^2}{C^2} < 1 \\ &\Leftrightarrow C^2 - 2AC + A^2 + B^2 < C^2 \Leftrightarrow C > \frac{A^2+B^2}{2A}. \end{aligned}$$

Then take $h \geq \frac{n-2}{\sqrt{C}}$ where C is large enough.

Hence a pair of complex zeros of $f(x)$, namely, that which corresponds to $A > 0$, can always be made stable by the choice of a sufficiently large h . By repeating the iterative process on each depressed polynomial, we can thus obtain all the complex zeros of $f(x)$.

Lemma 4. If $f(z)$ has real coefficients with all of its zeros v_j complex, then there exists a zero β whose imaginary part is positive such that $\operatorname{Re} \left(\prod_{v_j \neq \beta, \bar{\beta}} (v_j - \beta) \right) > 0$.

Proof.

$$\begin{aligned}
 \prod_{\gamma_j \neq \beta, \bar{\beta}} (\gamma_j - \beta) &= \prod_{\gamma_j \neq \beta, \beta, \bar{\beta}} (\beta - \gamma_j) \text{ since the degree } n \text{ is even,} \\
 &= \frac{f(z)}{(z - \beta)(z - \bar{\beta})} \Big|_{z=\beta} \\
 &= \frac{f(z) - f(\beta)}{z - \beta} \Big|_{z=\beta} \times \frac{1}{\beta - \bar{\beta}} \\
 &= \frac{f'(\beta)}{2i \operatorname{Im}(\beta)} = \frac{-2i f'(\beta)}{\operatorname{Im}(\beta)} \\
 &= \frac{-2i [\operatorname{Re}(f'(\beta)) + i \operatorname{Im}(f'(\beta))]}{\operatorname{Im}(\beta)}.
 \end{aligned}$$

$$\text{So } \operatorname{Re} \left(\prod_{\gamma_j \neq \beta, \bar{\beta}} (\gamma_j - \beta) \right) = \frac{2 \operatorname{Im}(f'(\beta))}{\operatorname{Im}(\beta)}.$$

Taking β such that $\operatorname{Im}(\beta) > 0$, $\operatorname{Re} \left(\prod_{\gamma_j \neq \beta, \bar{\beta}} (\gamma_j - \beta) \right) > 0$ if and only if

$\operatorname{Im}(f'(\beta)) > 0$. To show this is true for some β , consider the function $\frac{1}{f(z)}$, and the contour integral of the function around the curve C , composed of the real line and the upper semicircle. C , therefore, encloses half of the zeros of $f(z)$, namely, those whose imaginary parts are positive. By Cauchy's Residue Theorem, $\int_C dz/f(z) = 2\pi i \sum r_j$ where r_j stands for a residue of $1/f(z)$ at its pole inside C , and the summation taken over all such poles. Since the poles of $1/f(z)$ are precisely the zeros of $f(z)$ whose imaginary parts are positive,

$$r_j = \lim_{z \rightarrow \gamma_j} (z - \gamma_j) (1/f(z)) = \left[\lim_{z \rightarrow \gamma_j} \frac{f(z)}{z - \gamma_j} \right]^{-1} = \frac{1}{f'(\gamma_j)},$$

if γ_j is such a zero. Note that $f'(\gamma_j) \neq 0$ since γ_j is a simple zero of $f(z)$.

$$\begin{aligned}
\text{Therefore } \int_C dz/f(z) &= 2\pi i \sum_{\text{Im}(\gamma_j) > 0} 1/f'(\gamma_j) \\
&= 2\pi i \sum \frac{[\text{Re}(f'(\gamma_j)) - i\text{Im}(f'(\gamma_j))]}{|f'(\gamma_j)|^2} \\
&= 2\sum \left(\frac{\text{Im}(f'(\gamma_j))}{|f'(\gamma_j)|^2} \right) + 2\sum \left(\frac{\text{Re}(f'(\gamma_j))i}{|f'(\gamma_j)|^2} \right).
\end{aligned}$$

On the other hand,

$$\int_C dz/f(z) = \lim_{R \rightarrow \infty} \left[\int_{-R}^R dx/f(x) + \int_0^\pi \frac{\text{Re}^{i\varphi} i d\varphi}{f(\text{Re}^{i\varphi})} \right].$$

The first integral is a positive value since as $x \rightarrow \pm \infty$, $f(x) \rightarrow \infty$, using n even, while the second is 0 since the denominator of the integrand is dominated by R^n where $n > 1$. Hence $\int_C dz/f(z) > 0$. Comparing this with the first result, we deduce that $\sum \text{Im}(f'(\gamma_j)) > 0$, from which we further deduce that there is one, say β , of those γ_j 's such that $\text{Im}(f'(\beta)) > 0$. This proves the lemma.

Concluding Remarks

We have thus proved that Lin's Penultimate Remaindering can be made to converge to a quadratic factor by a proper choice of some positive number h . It would be very nice if h can be estimated which will work for all the zeros. However, this is very difficult to determine, if not impossible. For, even if we have all the estimates of the zeros we are seeking, still we have no way of finding out if a certain h is good enough to make a particular pair of zeros most stable, in the

sense that the said zeros would be the first ones to come out in the iterative process, because if they are not, then further translation might still be necessary on the depressed polynomial.

EXAMPLES

Example 1. $f(x) = x^4 - x^3 - x^2 - x - 2 = 0$

For $h = 0$, the following machine output is given for the coefficients of the divisor $x^2 + b(2)x + b(1)$:

b(1)	b(2)
2.0	1.0
2.0	-3.0
-0.6666667	-1.6666667
-2.5714287	-0.7142858
-1.1626864	-1.2686565
-2.4890507	-0.7554764
-1.5333674	-1.2333116
-2.4357176	-0.7821669
-1.5806082	-1.2096235
.....	

It is not converging.

With $h = 1.5$, the iteration converges although not rapidly:

$$f(x+h) = x^4 + 5x^3 + 8x^2 + 2.75x - 4.0625$$

-0.5078125	0.3437500
-0.5881521	0.7404567
-0.7475877	0.9670812
-0.8380733	1.1892824
-0.9434391	1.3803033
-1.0292210	1.5618733
-1.1101841	1.7185219
-1.1704483	1.8419024
-1.2114062	1.9222638
-1.2328593	1.9660164
-1.2431159	1.9860708
-1.2472219	1.9945306
-1.2489542	1.9978617
-1.2495781	1.9991815
-1.2498472	1.9996808

From the zeros of $x^2 + 1.9996808 - 1.2498472$ and the depressed polynomial, we obtain, after adding 1.5 to each of them, the zeros of $f(x)$:

$$\begin{array}{l} 2.0 \\ -0.99968 \\ -0.00016 \pm 0.99984i \end{array} \quad \begin{array}{l} \text{as compared to} \\ \text{the actual zeros} \end{array} \quad \left\{ \begin{array}{l} 2.0 \\ -1.0 \\ \pm i. \end{array} \right.$$

Example 2. $f(x) = x^4 - 2.075x^3 + .668x^2 - .091x + 2.28$

For $h = 0$, the following output shows divergence:

b(1)	b(2)
3.4131736	-0.1362275
-0.7576543	-2.1687413
1.3996702	-0.0122634
-3.0120230	-3.6938910
0.2360241	0.4953543
1.3370773	0.3024056
45.7249146	61.9246697
0.0005819	0.7468639
0.8216316	-0.0322014
.....	

Taking $h = 2$, so that $f(x+h) = x^4 + 5.925x^3 + 12.2180001x^2 + 9.681x + 4.17$, the iteration converges:

0.3412997	0.7923555
0.5339431	1.0152895
0.6224550	1.0537692
0.6452719	1.0288571
0.6380735	0.9979128
0.6258326	0.9810949
0.6185358	0.9770408
0.6163983	0.9786261
0.6167771	0.9809365
0.6176487	0.9822561
0.6182072	0.9826264

Thus, from the zeros of $x^2 + .9826264x + .618072$ and the depressed polynomial, we derive, after adding 2 to each of them, the zeros of the original polynomial, namely,

$$\begin{array}{l} 1.50869 \pm .61386 i \\ -0.47119 \pm .79782 i. \end{array}$$

Check: $\pi(x-r_1^*) = x^4 - 2.075x^3 + .6679878x^2 - 0.0904261x + 2.2776718.$

Remark: Since the coefficients of $\pi(x-r_1^*)$ differ from those of the original by significant amount, we can improve the zeros obtained here by applying the iteration in the next chapters.
(See Example 3 of Chapter IV.)

Example 3. $f(x) = x^3 - 8x^2 + 14x - 12$

For $h = 0$, trying for a linear divisor, the iteration fails to converge:

$b(1)$

-0.8571429
-1.5233160
-2.9027832
15.0731962
-0.0331687
-1.5453726
-2.9746884
12.6484070
.....

Taking $h = 5$, however, a linear divisor is obtained:

-1.8888889
-0.6591671
-1.2100789
-0.8978157
-1.0565054
-0.9707772
-1.0156622
-0.9917623
-1.0043762
-0.9976874
-1.0012255
-0.9993515
-1.0003434

The zeros of $f(x)$ are then found to be 6.00034 and $.99983 \pm 1.00086i$ (as compared to the actual zeros 6.0 and $1 \pm i$).

Example 4. $f(x) = x^6 + 256.8x^5 + 151.9x^4 + 258.5x^3 + 88.5x^2$
 $+ 4.5x + .05$

Applying Penultimate Remaindering without translation, two zeros, $-.01579$ and $-.0428$, are obtained after only three iterative stages. However, for the depressed polynomial $x^4 + 256.7414093x^3 + 136.8572083x^2 + 250.3082123x + 73.7423019$, the iteration diverges.

Applying then translation of variable with $h = 3$, the polynomial is transformed to $x^4 + 268.7414055x^3 + 2501.5298462x^2 + 8111.4692993x + 90693995361$, and the output for the coefficients of the quadratic divisor sought converges, though slowly:

b(1)	b(2)
3.6255412	3.2426034
5.5402665	4.3670773
6.7609104	4.9549237
7.6359290	5.3278508
8.3169955	5.5939965
8.8814500	5.8001015
9.3734536	5.9698291
9.8211602	6.1165862
10.2439576	6.2486506
10.6565874	6.3714687
11.0711598	6.4887165
11.4980766	6.6026727
11.9457625	6.7140132
12.4182427	6.8206988
12.9077997	6.9152735
13.3762462	6.9789250

The zeros of $x^2 + 6.9789250 + 13.3762462$ are $-3.4894625 \pm 1.0954i$ from which two zeros of the original polynomial, namely, $-.4894625 \pm 1.0954i$ are obtained.

The final depressed polynomial is $x^2 + 261.7624805x + 661.3328808$ whose zeros are -259.2111402 and -2.5513402 ; hence two zeros of $f(x)$ are -256.2111402 and $.4486598$.

Since we have used a factor whose coefficients have not converged yet, it is possible that the zeros obtained from it and consequently the succeeding zeros found may not be a close estimate of the actual zeros. Hence the iteration process which shall be established in the next two chapters can be used to improve those zeros. (See Example 10 of Chapter IV.)

Example 5. $f(x) = x^6 - 12x^5 + 50x^4 - 70x^3 - 41x^2 + 2x + 390$

As in previous examples, Penultimate Remaindering is divergent for $h = 0$.

For $h = 6$, $f(x)$ becomes $x^6 + 24x^5 + 230x^4 + 1130x^3 + 3019x^2 + 4046x + 1950$.

The following output shows convergence, but quite slowly:

b(1)	b(2)
0.6459092	1.3401789
1.1123276	1.9952449
1.4510382	2.4011633
1.7056451	2.6830334
1.9032140	2.8925371
2.0606095	3.0554308
2.1887029	3.1861302
2.2947577	3.2934612
2.3837912	3.3831297
2.4593760	3.4590352
2.5241261	3.5239491
2.5800025	3.5799099
2.6285094	3.6284607
2.6708241	3.6707982
2.7078851	3.7078715
2.7404516	3.7404443

At this point, instead of continuing the iteration, it might be wise to consider the above coefficients as those of a divisor of $f(x+h)$ with the intention of applying later as a corrective measure the iterative process in the next two chapters, since the latter converges quite fast when the input r_i are not too far off from the actual zeros.

Applying again Penultimate Remaindering on the depressed polynomial $x^4 + 20.259555x^3 + 151.4798088x^2 + 507.877880x + 704.1879913$:

4.6487251	3.3527761
7.8116007	4.7620618
10.0788074	5.5363917
11.7584547	6.0026574
13.0062971	6.2841839
13.9031010	6.4385376
14.4925766	6.4977605
14.8042977	6.4842725
14.8710879	6.4187196
14.7417496	6.2208961
14.4848443	6.3232256
14.1796360	6.1320639
13.8976722	6.0698298
13.6860651	6.0380195
13.5619417	6.0327021
13.5182898	6.0457873

Again, taking $x^2 + 6.0457873x + 13.5182898$ as divisor, the depressed polynomial is $x^2 + 14.2137684x + 52.0280985$. Using the quadratic formula on each of the divisors, the following zeros of $f(x)$ are determined:

$$\begin{aligned}
 -1.0000222 + 6 &= 4.9999778 \\
 -2.7404221 + 6 &= 3.2595779 \\
 -3.0228936 + 2.099291 + 6 &= 2.9771064 + 2.09291 \\
 -7.1068842 \pm 1.2331 + 6 &= -1.1068842 \pm 1.2331.
 \end{aligned}$$

As in the previous example, to improve these zeros, we will use them as input to the iteration which shall be established in the next chapters.

CHAPTER III

ESTIMATES OF THE ZEROS OF $g(x)$

For the purpose of estimating the zeros of $g(x)$, we propose to derive a set of formulas which will determine the values of η 's to be added to the known r_i 's to obtain the zeros r_i^* 's. These values can also serve as the first set of iterates in case the iterative procedure, which shall be discussed next, is to be employed to get better estimates of the desired zeros.

Case 1. $t(x)$ has a simple zero

Theorem 1. Let $r_i, i = 1, \dots, n$ be the n zeros of $t(x) = \sum_{i=1}^{n+1} a_i x^{i-1}$,

and $r_i^*, i = 1, \dots, n$ and r_{n+1}^* be those of

$$g(x) = \alpha_{n+2} x^{n+1} + \sum_{i=1}^{n+1} (a_i + \alpha_i) x^{i-1}.$$

Writing $r_i^* = r_i + \eta_i$ where $\eta_i \rightarrow 0$ as the α_q 's $\rightarrow 0$, for $i = 1, \dots, n$,

and $r_{n+1}^* = -1/\alpha_{n+2} + \eta_{n+1}$, then

$$(4) \quad \eta_k \approx \frac{\sum_{q=1}^{n+2} \alpha_q r_k^{q-1}}{\sum_{i \neq k} (r_k - r_i)} \quad \text{for all } k \text{ such that } r_k \text{ is a simple zero.}$$

$$(5) \quad \eta_{n+1} \approx \frac{n}{- \sum_{i=1}^n r_i + \alpha_n - \alpha_{n+2}} \left[\sum_{\substack{w < i \\ i, w=1}}^n r_i r_w - \left(\sum_{i=1}^n r_i \right)^2 \right]$$

α_{n+1} may be set to 0 by assuming $t(x)$ and $f(x) = g(x) - \alpha_{n+2} x^{n+1}$ to be monic polynomials.

Proof. $g(x) = \alpha_{n+2} \prod_{i=1}^n (x - r_i^*)(x - r_{n+1}^*) = \sum \alpha_q x^{q-1} + \pi(x - r_1) \Rightarrow$

$$(6) \quad \alpha_{n+2} \prod_{i=1}^n (x - r_i - \eta_i)(x + \frac{1}{\alpha_{n+2}} - \eta_{n+1}) = \\ = \sum \alpha_q x^{q-1} + \pi(x - r_1)$$

where from here on, i ranges from 1 to n ; q , from 1 to $n+2$, unless otherwise specified.

To solve for η_k , let $x = r_k$ in (6):

$$(-\eta_k) \prod_{i \neq k} (r_k - r_i - \eta_i) (\alpha_{n+2} r_k^{n+1} - \alpha_{n+2} \eta_{n+1}) = \sum \alpha_q r_k^{q-1} \Rightarrow$$

$$(7) \quad \eta_k = \frac{-\sum \alpha_q r_k^{q-1}}{\prod_{i \neq k} (r_k - r_i - \eta_i) [1 + \alpha_{n+2} (r_k - \eta_{n+1})]} \\ = \frac{-\sum \alpha_q r_k^{q-1}}{\prod_{i \neq k} (r_k - r_i) [1 - \frac{\eta_i}{r_k - r_i}] [1 + \alpha_{n+2} (r_k - \eta_{n+1})]}$$

Expanding the denominator as power series and neglecting terms in α_q 's and η_i 's of order greater than 1,

$$\eta_k = \frac{-\sum \alpha_q r_k^{q-1}}{\prod_{i \neq k} (r_k - r_i)} \prod_{i \neq k} [1 + \frac{\eta_i}{r_k - r_i} + \frac{\eta_i^2}{(r_k - r_i)^2} + \dots] \times \\ [1 - \alpha_{n+2} (r_k - \eta_{n+1}) + \alpha_{n+2}^2 (r_k - \eta_{n+1})^2 - \dots] \\ \approx \frac{-\sum \alpha_q r_k^{q-1}}{\prod_{i \neq k} (r_k - r_i)} \text{ assuming } |\alpha_{n+2} (r_k - \eta_{n+1})| \text{ and } \left| \frac{\eta_i}{(r_k - r_i)} \right| \text{ are}$$

each less than 1, which are valid for α_q 's sufficiently small, since

$\eta_i \rightarrow 0$ with α_q 's, and from (6) by setting $x = 0$,

$$\alpha_{n+2} \eta_{n+1} = 1 - \frac{(-1)^n \alpha_1 + \pi r_1}{\pi(r_1 + \eta_1)}$$

which $\rightarrow 0$ with α_q 's.

To obtain (5), set $x = -1/\alpha_{n+2}$ in (6). Then

$$\frac{(-1)^{n+1}}{\alpha_{n+2}^n} + \frac{(-1)^n \alpha_{n+1}}{\alpha_{n+2}^n} + \frac{(-1)^{n-1} \alpha_n}{\alpha_{n+2}^{n-1}} + \dots \alpha_1$$

$$+ \frac{\pi(1+r_1 \alpha_{n+2})(-1)^n}{\alpha_{n+2}^n} = \alpha_{n+2}(-\eta_{n+1})(-1)^n \frac{\pi[1+(r_1+\eta_1)\alpha_{n+2}]}{\alpha_{n+2}^n}.$$

Multiplying by $\alpha_{n+2}^n (-1)^n$ and dropping the term in α_{n+1} since

$\alpha_{n+1} = 0$ from $f(x)$ and $t(x)$ monic,

$$(8) \quad \eta_{n+1} = \frac{-[-1 + \sum_{u=1}^n (-1)^u \alpha_{n+1-u}(\alpha_{n+2})^u + \pi(1+r_1 \alpha_{n+2})]}{\alpha_{n+2} \pi[1 + \alpha_{n+2}(r_1 + \eta_1)]}.$$

Expanding the denominator as power series and dropping powers in

α_q 's and η_i 's in the numerator of order greater than 2,

$$\eta_{n+1} \approx \frac{-1}{\alpha_{n+2}} [-1 - \alpha_n \alpha_{n+2} + 1 + \sum r_i \alpha_{n+2} + \sum_{w<1} \sum r_i r_w \alpha_{n+2}^2] x$$

$$\pi[1 - \alpha_{n+2}(r_1 + \eta_1) + \alpha_{n+2}^2 r_1^2]$$

$$\approx \frac{-1}{\alpha_{n+2}} [-\alpha_n \alpha_{n+2} + \sum r_i \alpha_{n+2} + \sum_{w<1} \sum r_i r_w \alpha_{n+2}^2][1 - \sum r_i \alpha_{n+2}]$$

$$\approx [\alpha_n - \sum r_i - \sum_{w<1} \sum r_i r_w \alpha_{n+2} + (\sum r_i)^2 \alpha_{n+2}].$$

Case 2. $t(x)$ has a multiple zero

Theorem 2. Let r be a zero of multiplicity $n_1 > 1$, and r_s ,

$s = n_1 + 1, \dots, n$, be the remaining zeros. Let r_t^* ,

$t = 1, \dots, n_1$; r_s^* , $s = n_1 + 1, \dots, n$; and r_{n+1}^* be the

$n+1$ zeros of $g(x)$, where $r_t^* = r + \eta_t$, $r_s^* = r_s + \eta_s$, and

$r_{n+1}^* = \frac{-1}{\alpha_{n+2}} + \eta_{n+1}$ and $g(x)$ is as previously defined.

Then a monic polynomial $\sum_0^{n_1} C_{k+1} y^k$ of degree n_1 having the η_t 's as

zeros can be constructed where the coefficients C_{k+1} are determined by

$$(9) \quad C_{k+1} = \frac{1}{\frac{n}{k!} \prod_{s=1}^{n_1+1} (r-r_s)} \left\{ \sum_{q=k}^{n+1} \frac{\alpha_{q+1} q! r^{q-k}}{(q-k)!} - \sum_{m=1}^k \binom{k}{m} \pi^{[m]}(x-r_s)_{x=r} x^{k-m} \right\}.$$

Here $\pi^{[m]}(x-r_s)_{x=r}$ denotes the m th derivative of $(x-r_s)$ with respect to x , evaluated at $x=r$.

Proof.

A polynomial $\sum_0^{n_1} C_{k+1} y^k$ having the η_t 's as zeros must

have coefficients C_{k+1} satisfying the relation $C_{k+1} = (-1)^{n_1-k}$

multiplied by the sum of the products of the η_t 's taken (n_1-k) at

a time, $k = 0, \dots, n_1-1$ and $C_{n_1+1} = 1$. Indeed, this is the value of

$$(1/k!) \pi_t^{[k]}(r) \text{ where } \pi_t^{[k]}(r) \stackrel{\text{def.}}{=} \frac{d^k}{dx^k} \left[\pi_{t=1}^{n_1} (x-r-\eta_t) \right]$$

evaluated at $x=r$.

$$g(x) = \sum_{q=0}^{n+1} \alpha_{q+1} x^q + (x-r)^{n_1} \pi_{s=n_1+1}^n (x-r_s) \Rightarrow$$

$$(10) \quad g^{[k]}(r) = \sum_{q=k}^{n+1} \alpha_{q+1} q! r^{q-k} / (q-k)!$$

$$\text{But from } g(x) = \alpha_{n+2}^* (x-r_{n+1}^*) \pi_{(x-r_1-\eta_1)}^{(x-r_{n+1}^*)}$$

$$= \alpha_{n+2} \left(x + \frac{1}{\alpha_{n+2}} - \eta_{n+1} \right) \pi_t(x) \pi_s(x), \text{ where from}$$

here on, t ranges from 1 to n_1 ; s , from n_1+1 to n , and as in the previous theorem, i from 1 to n , and q from 1 to $n+2$, unless otherwise

$$\text{specified; } \pi_t^{(x)} \stackrel{\text{def.}}{=} \pi(x-r-\eta_t) \text{ and } \pi_s^{(x)} \stackrel{\text{def.}}{=} \pi(x-r_s-\eta_s),$$

we can proceed by induction to show that

$$(11) \quad g^{[k]}(x) = \alpha_{n+2}^k \sum_{m=0}^{k-1} \binom{k-1}{m} \pi_t^{[k-1-m]}(x) \pi_s^{[m]}(x) +$$

$$(\alpha_{n+2}^x + 1 - \alpha_{n+2} \eta_{n+1}) \sum_{m=0}^k \binom{k}{m} \pi_t^{[k-m]}(x) \pi_s^{[m]}(x).$$

It may easily be verified for $k = 0, 1$, so assuming the identity true for k , we can show it true for $k+1$ as follows:

$$[g^{[k]}(x)]' = \alpha_{n+2}^k \sum_{m=0}^{k-1} \binom{k-1}{m} [\pi_t^{[k-m]}(x) \pi_s^{[m]}(x) + \pi_t^{[k-m-1]}(x) x$$

$$\pi_s^{[m+1]}(x)] + [\alpha_{n+2}^x + 1 - \alpha_{n+2} \eta_{n+1}] \sum_{m=0}^k \binom{k}{m} \pi_t^{[k+1-m]}(x) x$$

$$\pi_s^{[m]}(x) + \pi_t^{[k-m]}(x) \pi_s^{[m+1]}(x)] + \alpha_{n+2} \sum_{m=0}^k \binom{k}{m} \pi_t^{[k-m]}(x) \pi_s^{[m]}(x)$$

$$= \alpha_{n+2}^k \sum_{m=0}^k \binom{k}{m} \pi_t^{[k-m]}(x) \pi_s^{[m]}(x) + [1 + \alpha_{n+2}^{(x-\eta)_{n+1}}] x$$

$$\sum_{m=0}^{k+1} \binom{k+1}{m} \pi_t^{[k-m+1]}(x) \pi_s^{[m]}(x)$$

$$+ \alpha_{n+2}^k \sum_{m=0}^k \binom{k}{m} \pi_t^{[k-m]}(x) \pi_s^{[m]}(x)$$

using the lemma below on the first two groups of summation in the derivative, with $v = k$ and $k + 1$, respectively.

Lemma. $\sum_{m=0}^{v-1} \binom{v-1}{m} [\pi_t^{[v-m]}(x) \pi_s^{[m]}(x) + \pi_t^{[v-1-m]}(x) \pi_s^{[m+1]}(x)] =$

$$\sum_{m=0}^v \binom{v}{m} \pi_t^{[v-m]}(x) \pi_s^{[m]}(x) \quad \text{for any positive integer } v.$$

Proof. Left-hand side =

$$\sum_{m=0}^{v-1} \binom{v-1}{m} \pi_t^{[v-m]}(x) \pi_s^{[m]}(x) + \sum_{m=0}^{v-1} \binom{v-1}{m} \pi_t^{[v-1-m]}(x) \pi_s^{[m+1]}(x) =$$

$$\sum_{m=0}^{v-1} \binom{v-1}{m} \pi_t^{[v-m]}(x) \pi_s^{[m]}(x) + \sum_{m=1}^v \binom{v-1}{m-1} \pi_t^{[v-m]}(x) \pi_s^{[m]}(x) =$$

$$\sum_{m=1}^{v-1} \left[\binom{v-1}{m} + \binom{v-1}{m-1} \right] \pi_t^{[v-m]}(x) \pi_s^{[m]}(x) + \binom{v-1}{0} \pi_t^{[v]}(x) \pi_s^{[0]}(x) +$$

$$\binom{v-1}{v-1} \pi_t^{[0]}(x) \pi_s^{[v]}(x)$$

$$= \sum_{m=1}^{v-1} \binom{v}{m} \pi_t^{[v-m]}(x) \pi_s^{[m]}(x) + \binom{v}{0} \pi_t^{[v]}(x) \pi_s^{[0]}(x) + \binom{v}{v} \pi_t^{[0]}(x) \pi_s^{[v]}(x) =$$

= right-hand side. (QED)

Now, equating (10) and (11) evaluated at r ,

$$\sum_{q=k}^{n+1} \frac{\alpha_{q+1} q! r^{q-k}}{(q-k)!} = \alpha_{n+2} k \sum_{m=0}^{k-1} \binom{k-1}{m} \pi_t^{[k-1-m]}(r) \pi_s^{[m]}(r) +$$

$$[\alpha_{n+2}^{r+1-\alpha_{n+2}} \eta_{n+1}] \sum_{m=0}^k \binom{k}{m} \pi_t^{[k-m]}(r) \pi_s^{[m]}(r).$$

Solving for $\pi_t^{[k]}(r)$ ($m=0$ of the last summation),

$$(12) \pi_t^{[k]}(r) = \left\{ \sum_{q=k}^{n+1} \alpha_{q+1} q! r^{q-k} / (q-k)! - \alpha_{n+2} k \sum_{m=0}^{k-1} \binom{k-1}{m} \pi_t^{[k-1-m]}(r) \times \right.$$

$$\left. \pi_s^{[m]}(r) \right\} / [1 + \alpha_{n+2} (r - \eta_{n+1})] - \sum_{m=1}^k \binom{k}{m} \pi_t^{[k-m]}(r) \pi_s^{[m]}(r) \Bigg] / \pi_s(r).$$

Expanding

$$1/\pi(r) = 1/\pi(r-r_s) [1 - \eta_s / (r-r_s)] \text{ and } 1/[1 + \alpha_{n+2} (r - \eta_{n+1})]$$

as power series and neglecting terms in α_q 's, η_1 's and of C_{v+1} 's

$$\pi_t^{[v]}(r) \quad \left(\text{i.e., } \frac{t}{v!} \text{ 's of order greater than 1, we arrive finally at} \right.$$

$$\pi_t^{[k]}(r) \simeq \frac{\sum_{q=k}^{n+1} \frac{\alpha_{q+1} q! r^{q-k}}{(q-k)!} - \sum_{m=1}^k \binom{k}{m} \pi_{(x-r_s)}^{[m]} C_{k-m+1} (k-m)!}{\pi(r - r_s)}$$

From this, (9) follows immediately.

CHAPTER IV

A NONCONTAMINATING ITERATIVE PROCESS FOR DETERMINING THE ZEROS OF $g(x)$

Case 1. $t(x)$ has simple zeros

Theorem 1. Let $t(x)$ have zeros r_i , $i = 1, \dots, n$. If r_i^* 's are the corresponding zeros of $g(x)$, that is, $r_i^* = r_i + \eta_i$, $i = 1, \dots, n$ such that $\eta_i \rightarrow 0$ with the α_q 's, and $r_{n+1}^* = -1/\alpha_{n+2} + \eta_{n+1}$, then $r_k + \eta_k^{(j)} \rightarrow r_k^*$ as $j \rightarrow \infty$ for all k such that r_k is simple zero of $t(x)$, and $-1/\alpha_{n+2} + \eta_{n+1}^{(j)} \rightarrow r_{n+1}^*$, where

$$(13) \quad \eta_k^{(j+1)} = \frac{-\sum_{q=1}^n \alpha_q r_k^{q-1}}{\prod_{l \neq k} (r_k - r_l - \eta_l^{(j)}) [1 + \alpha_{n+2} (r_k - \eta_{n+1}^{(j)})]} \quad j > 0$$

$$(14) \quad \eta_{n+1}^{(j+1)} = \frac{1 - \sum_{u=1}^n (-1)^u \alpha_{n+1-u} (\alpha_{n+2})^u - \pi(1 + r_1 \alpha_{n+2})}{\alpha_{n+2} \prod_{i=1}^n [1 + \alpha_{n+2} (r_i + \eta_i^{(j)})]}$$

and $\eta_k^{(1)}$ and $\eta_{n+1}^{(1)}$ are the estimates (4) and (5), respectively, of the previous chapter.

Proof. Comparing (13) and (14) with (7) and (8), respectively, of the previous chapter, we see clearly how the iteration formulas are arrived at, namely, by replacing all the η 's in the right-hand side of

(8) and (7) by $\eta^{(j)}$'s and those at the left-hand side by $\eta^{(j+1)}$'s.

Let us defer the proof of the convergence of these iterative procedures for sufficiently small α_q 's, and go to Case 2.

Case 2. $t(x)$ has a multiple zero

Theorem 2. Let r be a zero of multiplicity $n_1 > 1$ of $t(x)$. Let r_t^* ,

$t = 1, \dots, n_1$ be the corresponding zeros of $g(x)$, i.e.,

$r_t^* = r + \eta_t$ where $\eta_t \rightarrow 0$ with the α_q 's, and r_s , $s = n_1 + 1, \dots, n$

be the remaining zeros of $t(x)$. Then $r + \eta_t^{(j)} \rightarrow r_t^*$ as $j \rightarrow \infty$

where $\eta_t^{(j)}$'s are the zeros of the monic polynomial $\sum_{k=0}^{n_1} C_{k+1}^{(j)} y^k$,

the coefficients $C_{k+1}^{(j)}$ being determined by

$$(15) \quad C_{k+1}^{(j+1)} = \begin{cases} \frac{n_1}{\sum_{q=k}^{n_1} \frac{\alpha_{q+1} q! r^{q-k}}{(q-k)!}} - \alpha_{n+2}^k \sum_0^{k-1} \binom{k-1}{m} (k-1-m)! C_{k-m}^{(j+1)} x \\ (j > 0) \end{cases}$$

($j > 0$)

$$\pi^{[m]}(x - r_s - \eta_s^{(j)})_{x=r} - [1 + \alpha_{n+2}^{(j)}(r - \eta_{n+1}^{(j)})] x$$

$$\left. \sum_{m=1}^k \binom{k}{m} (k-m)! C_{k+1-m}^{(j+1)} \pi^{[m]}(x - r_s - \eta_s^{(j)})_{x=r} \right\} /$$

$$k! \pi(r - r_s - \eta_s^{(j)}) [1 + \alpha_{n+2}^{(j)}(r - \eta_{n+1}^{(j)})] \text{ where as}$$

stated before, i ranges from 1 to n and s from $n_1 + 1$ to n , while

$C_{k+1}^{(1)}$'s are the estimates (9) obtained in the previous chapter.

Proof. As was already noted in the previous chapter, C_{k+1} is precisely $\pi_t^{[k]}(r)/k!$; comparing (12) with (15) above, the derivation of the iteration formula is clear.

We shall now establish the convergence of (13), (14), and (15) simultaneously for sufficiently small values α_q 's. Without loss of generality, let us, for the sake of simplicity, assume that $t(x)$ has one multiple zero r to which corresponds r_t^* 's.

We shall presently aim to obtain a matrix equation

$E^{(j+1)} = M E^{(j+1)} + N E^{(j)}$ where M and N are $(n+1)$ by $(n+1)$ matrices to be determined and $E^{(j)}$ is a column vector with elements $E_k^{(j)} = C_k^{(j)} - C_k$ for $k = 1, \dots, n_1$ while $E_k^{(j)} = \eta_k^{(j)} - \eta_k$ for $k = n_1 + 1, \dots, n + 1$.

Let us then go back and consider Case 1. From (7) and (13), for $k = n_1 + 1, \dots, n$,

$$\begin{aligned}
 (16) \quad E_k^{(j+1)} &= - \sum_{q=1}^{n+2} \alpha_q r_k^{q-1} \left\{ \prod_{i \neq k} (r_k - r_i - \eta_i) [1 + \alpha_{n+2} (r_k - \eta_{n+1})] - \right. \\
 &\quad \left. \prod_{i \neq k} (r_k - r_i - \eta_i^{(j)}) [1 + \alpha_{n+2} (r_k - \eta_{n+1} - E_{n+1}^{(j)})] \right\} / \\
 &\quad \prod_{i \neq k} (r_k - r_i - \eta_i^{(j)}) \prod_{i \neq k} (r_k - r_i - \eta_i) [1 + \alpha_{n+2} (r_k - \eta_{n+1})] \times \\
 &\quad [1 + \alpha_{n+2} (r_k - \eta_{n+1} - E_{n+1}^{(j)})] \\
 &= - \sum_q \alpha_q r_k^{q-1} \left\{ \prod_{i \neq k} (r_k - r_i - \eta_i) [1 + \alpha_{n+2} (r_k - \eta_{n+1})] - \right. \\
 &\quad \left. \prod_{s \neq k} (r_k - r_s - \eta_s - E_s^{(j)}) [\sum (r_k - r)^v C_{v+1} + \sum_{0}^{n_1-1} (r_k - r)^v E_{v+1}^{(j)}] \right\} \times \\
 &\quad [1 + \alpha_{n+2} (r_k - \eta_{n+1} - E_{n+1}^{(j)})] \Bigg/ \prod_{i \neq k} (r_k - r_i - \eta_i) \times
 \end{aligned}$$

$$[\sum (r_k - r)^v C_{v+1} + \sum_0^{n_1-1} (r_k - r)^v E_{v+1}^{(j)}] \times$$

$$[\sum_{s \neq k} (\pi (r_k - r_s - \eta_s) - E_s^{(j)})] \times$$

$$[1 + \alpha_{n+2}(r_k - \eta_{n+1})][1 + \alpha_{n+2}(r_k - \eta_{n+1} - E_{n+1}^{(j)})],$$

where from now on, v ranges from 0 to n_1 , unless otherwise specified.

Neglecting terms in $E^{(j)}$'s of order > 1 , numerator

$$N \approx -\sum_q \alpha_q r_k^{q-1} \left\{ \sum_{i \neq k} (\pi (r_k - r_i - \eta_i) [1 + \alpha_{n+2}(r_k - \eta_{n+1})] - \right.$$

$$[\sum_{s \neq k} (\pi (r_k - r_s - \eta_s) - \sum_{p \neq k} E_p^{(j)} (\sum_{s \neq p, k} (\pi (r_k - r_s - \eta_s)))] \times$$

$$[\sum (r_k - r)^v C_{v+1} + \sum_0^{n_1-1} (r_k - r)^v E_{v+1}^{(j)}] \times$$

$$[1 + \alpha_{n+2}(r_k - \eta_{n+1}) - \alpha_{n+2} E_{n+1}^{(j)}] \left\{ \right.$$

where from here on, p ranges from n_1+1 to n unless otherwise specified, and empty product is defined 1.

Expanding and neglecting further terms in $E^{(j)}$'s of order > 1 , the terms independent of $E^{(j)}$'s cancel out, leaving

$$(17) N \approx \sum_q \alpha_q r_k^{q-1} \left\{ [\sum (r_k - r)^v C_{v+1}] (1 + \alpha_{n+2}(r_k - \eta_{n+1})) \sum_{p \neq k} E_p^{(j)} \times \right.$$

$$(\sum_{s \neq p, k} (\pi (r_k - r_s - \eta_s)) - \sum_{s \neq k} (\pi (r_k - r_s - \eta_s) [1 + \alpha_{n+2}(r_k - \eta_{n+1})]) \times$$

$$\left. \sum_0^{n_1-1} (r_k - r)^v E_{v+1}^{(j)} + \alpha_{n+2} \sum_{s \neq k} (\pi (r_k - r_s - \eta_s) [\sum (r_k - r)^v C_{v+1}] E_{n+1}^{(j)}) \right\}.$$

Similarly, denominator

$$D \approx \prod_{i \neq k}^n (r_k - r_i - \eta_i) [1 + \alpha_{n+2}(r_k - \eta_{n+1})]^2 \left\{ \left[1 + \frac{\sum_{v=0}^{n_1-1} (r_k - r)^v E_{v+1}^{(j)}}{\sum (r_k - r)^v C_{v+1}} \right] \times \right. \\ \left. \left[1 - \frac{\sum E_p^{(j)} \left(\prod_{s \neq p, k} (r_k - r_s - \eta_s) \right)}{\prod_{s \neq k} (r_k - r_s - \eta_s)} \right] \left[1 - \frac{\alpha_{n+2} E_{n+1}^{(j)}}{1 + \alpha_{n+2}(r_k - \eta_{n+1})} \right] \right\}.$$

Expanding the reciprocal of each factor within the brace as power series, and noting that the numerator has no term independent of $E^{(j)}$'s, we only need the constant of the resulting expansion, namely, 1, for orders of $E^{(j)}$'s > 1 are being ignored. Thus

$$(18) D \approx \prod_{i \neq k}^n (r_k - r_i - \eta_i) [1 + \alpha_{n+2}(r_k - \eta_{n+1})]^2.$$

Varying k from n_1+1 to n , we thus derive from (17) and (18) the (n_1+1) st to n th row elements of our desired matrices M and N . Note that the (n_1+1) st to n th rows of M are filled with 0's.

Moreover, from (8) and (14),

$$E_{n+1}^{(j+1)} = \frac{-1 + \sum_{u=1}^n (-\alpha_{n+2})^u \alpha_{n-u+1} + \pi(1+r_i \alpha_{n+2})}{\alpha_{n+2}} \times \\ \left\{ \frac{\pi[1 + \alpha_{n+2}(r_i + \eta_i)^{(j)}] - \pi[1 + \alpha_{n+2}(r_i + \eta_i)]}{\pi[1 + \alpha_{n+2}(r_i + \eta_i)^{(j)}] \pi[1 + \alpha_{n+2}(r_i + \eta_i)]} \right\}.$$

Numerator N within the brace

$$= \pi [1 + \alpha_{n+2}(r_s + \eta_s) + \alpha_{n+2} E_s^{(j)}] [\Sigma (-\alpha_{n+2})^{n_1-v} (1 + \alpha_{n+2} r)^v C_{v+1} \\ + \sum_0^{n_1-1} (-\alpha_{n+2})^{n_1-v} (1 + \alpha_{n+2} r)^v E_{v+1}^{(j)}] - \pi [1 + \alpha_{n+2}(r_i + \eta_i)].$$

Neglecting terms in $E^{(j)}$'s of order > 1 , and combining terms, we get

$$N \approx \alpha_{n+2} [\Sigma (-\alpha_{n+2})^{n_1-v} (1 + \alpha_{n+2} r)^v C_{v+1}] \Sigma E_p^{(j)} \left(\frac{\pi}{s \neq p} (1 + \alpha_{n+2}(r_s + \eta_s)) \right) + \\ + \pi [1 + \alpha_{n+2}(r_s + \eta_s)] \sum_0^{n_1-1} (-\alpha_{n+2})^{n_1-v} (1 + \alpha_{n+2} r)^v E_{v+1}^{(j)}.$$

Similarly, denominator D within the brace

$$\approx \pi^2 [1 + \alpha_{n+2}(r_i + \eta_i)]. \text{ Hence}$$

$$(19) E_{n+1}^{(j+1)} \approx \frac{-1 + \sum_1^n (-\alpha_{n+2})^u \alpha_{n-u+1} + \pi (1 + r_i \alpha_{n+2})}{\alpha_{n+2} \pi^2 [1 + \alpha_{n+2}(r_i + \eta_i)]} \left\{ \sum_0^{n_1-1} (-\alpha_{n+2})^{n_1-v} (1 + \alpha_{n+2} r)^v x \right. \\ \left. E_{v+1}^{(j)} + \alpha_{n+2} [\Sigma (-\alpha_{n+2})^{n_1-v} (1 + \alpha_{n+2} r)^v C_{v+1}] x \right. \\ \left. \Sigma E_p^{(j)} \left(\frac{\pi}{s \neq p} (1 + \alpha_{n+2}(r_s + \eta_s)) \right) \right\}$$

from which we get the last row elements of M and N. Note that (19) does not also give any nonzero contribution to the matrix M. Note further that each term of the numerator in the expanded form of (19) has at least an order -2 factor of α_q 's, so that each coefficient of $E_{v+1}^{(j)}$ and $E_p^{(j)}$ which is an element of the matrix N goes to 0 with the α_q 's.

Leaving this for the meantime, let us obtain for Case 2 an analogous expression for $E_k^{(j+1)}$ when $k = 1, \dots, n_1$. From (15) and its corresponding expression for C_{k+1} , for $k = 0, 1, \dots, n_1 - 1$,

$$\begin{aligned}
 E_{k+1}^{(j+1)} = & \left\{ \sum_{q=k}^{n+1} \alpha_{q+1} r^{q-k} q! / (q-k)! - \alpha_{n+2}^k \sum_0^{k-1} \binom{k-1}{m} (k-m-1)! C_{k-m}^{(j+1)} \right\} x \\
 & \pi^{[m]}(x - r_s - \eta_s^{(j)})_{x=r} - [1 + \alpha_{n+2}^{(r-\eta_{n+1})}] \sum_1^k \binom{k}{m} x \\
 & (k-m)! C_{k+1-m}^{(j+1)} \pi^{[m]}(x - r_s - \eta_s^{(j)})_{x=r} \left\{ \pi(r - r_s - \eta_s^{(j)}) \right. \\
 & [1 + \alpha_{n+2}^{(r-\eta_{n+1})}] - \left. \left\{ \sum_{q=k}^{n+1} \alpha_{q+1} r^{q-k} q! / (q-k)! - \right. \right. \\
 & \left. \left. \alpha_{n+2}^k \sum_0^{k-1} \binom{k-1}{m} (k-1-m)! C_{k-m}^{[m]} \pi^{[m]}(x - r_s - \eta_s^{(j)})_{x=r} - \right. \right. \\
 & \left. \left. [1 + \alpha_{n+2}^{(r-\eta_{n+1})}] \sum_1^k \binom{k}{m} (k-m)! C_{k+1-m}^{[m]} \pi^{[m]}(x - r_s - \eta_s^{(j)})_{x=r} \right\} x \right. \\
 & \left. \pi(r - r_s - \eta_s^{(j)}) [1 + \alpha_{n+2}^{(r-\eta_{n+1})}] \right\} / k! \pi(r - r_s - \eta_s^{(j)}) x \\
 & \pi(r - r_s - \eta_s^{(j)}) [1 + \alpha_{n+2}^{(r-\eta_{n+1})}] [1 + \alpha_{n+2}^{(r-\eta_{n+1})}^{(j)}] \}.
 \end{aligned}$$

Proceeding in the same manner as in the previous case, that is, expanding out and neglecting powers of $E_k^{(j)}$ higher than 1, the terms independent of $E_k^{(j)}$'s drop out, leaving finally a linear combination of the $E_k^{(j)}$'s, namely,

$$\begin{aligned}
(20) \quad E_{k+1}^{(j+1)} &\approx \left\{ - \pi(r-r_s-\eta_s)[1+\alpha_{n+2}(r-\eta_{n+1})] \alpha_{n+2}^k \sum_0^{k-1} \binom{k-1}{m} x \right. \\
&\quad (k-1-m)! \pi^{[m]}(x-r_s-\eta_s)_{x=r} E_{k-m}^{(j+1)} - \\
&\quad \pi(r-r_s-\eta_s)[1+\alpha_{n+2}(r-\eta_{n+1})]^2 \sum_1^k \binom{k}{m} (k-m)! x \\
&\quad \pi^{[m]}(x-r_s-\eta_s)_{x=r} E_{k+1-m}^{(j+1)} + \pi(r-r_s-\eta_s) x \\
&\quad [1+\alpha_{n+2}(r-\eta_{n+1})] \alpha_{n+2}^k \sum_0^{k-1} \binom{k-1}{m} (k-1-m)! C_{k-m} x \\
&\quad [\sum_p E_p^{(j)}]_{s \neq p} \pi^{[m]}(x-r_s-\eta_s)_{x=r} + \pi(r-r_s-\eta_s) x \\
&\quad [1+\alpha_{n+2}(r-\eta_{n+1})]^2 \sum_1^k \binom{k}{m} (k-m)! C_{k+1-m} [\sum_p E_p^{(j)}] x \\
&\quad (\pi_{s \neq p}^{[m]}(x-r_s-\eta_s)_{x=r}) + \{[1+\alpha_{n+2}(r-\eta_{n+1})] x \\
&\quad [\sum_{q=k}^{n+1} \alpha_{q+1} r^{q-k} q! / (q-k)! - \alpha_{n+2}^k \sum_0^{k-1} \binom{k-1}{m} (k-1-m)! x \\
&\quad C_{k-m} \pi^{[m]}(x-r_s-\eta_s)_{x=r}] - [1+\alpha_{n+2}(r-\eta_{n+1})]^2 x \\
&\quad \sum_1^k \binom{k}{m} (k-m)! C_{k+1-m} \pi^{[m]}(x-r_s-\eta_s)_{x=r} \} \sum_p E_p^{(j)} x \\
&\quad (\pi_{s \neq p}(r-r_s-\eta_s)) + \alpha_{n+2} \pi(r-r_s-\eta_s) [\sum_{q=k}^{n+1} \alpha_{q+1} r^{q-k} q! /
\end{aligned}$$

$$(q-k)! - \alpha_{n+2}^k \sum_{0}^{k-1} \binom{k-1}{m} (k-1-m)! C_{k-m} \pi^{[m]}(x$$

$$-r_s - \eta_s)_{x=r} E_{n+1}^{(j)} \Bigg/ k! \pi^2 (r - r_s - \eta_s) x$$

$$[1 + \alpha_{n+2} (r - \eta_{n+1})]^2.$$

Varying k from 0 to $n_1 - 1$ in (20) we can get the first n_1 row elements of the matrices M and N . This completes our determination of the matrices M and N .

Note that in the vector equation $E^{(j+1)} = M E^{(j+1)} + N E^{(j)}$, the nonzero elements of M are found only in the first n_1 rows to the left of an excluding the diagonal. As for N , note that each of its elements has in every term either a factor α_q or C_k of power greater than 0 and since $C_k \rightarrow 0$ with the α_q 's, together with the fact that the η 's - finite number, we can make each of its elements arbitrarily small with the α_q 's.

To illustrate it for $n = 4$ and $n_1 = 3$, see the following pages.

M = -	$\begin{array}{c} 0 \\ \frac{\alpha_6}{[1+\alpha_6(r-\eta_5)]} + \frac{1}{[r-r_4-\eta_4]} \\ \frac{2\alpha_6}{[r-r_4-\eta_4][1+\alpha_6(r-\eta_5)]} \\ 0 \\ 0 \end{array}$	$\begin{array}{c} 0 \\ 0 \\ \frac{2\alpha_6}{[1+\alpha_6(r-\eta_5)]} + \frac{2}{r-r_4-\eta_4} \\ 0 \\ 0 \end{array}$	
	$\begin{array}{c} 0 \end{array}$	$\begin{array}{c} 0 \end{array}$	
	$\begin{array}{c} 0 \end{array}$	$\begin{array}{c} 0 \end{array}$	
N =	$\begin{array}{c} 0 \end{array}$	$\begin{array}{c} 0 \end{array}$	
	$\begin{array}{c} \frac{6}{-1+\sum_1 \alpha_q} r_4^{q-1} \frac{3}{1} (r_4-r_1-\eta_1)^2 [1+ \\ \alpha_6(r_4-\eta_5)] \end{array}$	$\begin{array}{c} \frac{6}{-(r_4-r_1-\eta_1)^2 [1+\alpha_6(r_4-\eta_5)]} \frac{3}{1} (r_4-r_1-\eta_1)^2 [1+ \\ \alpha_6(r_4-\eta_5)] \end{array}$	$\begin{array}{c} (r_4-r_1)^2 (\sum \alpha_6 \\ -\eta_1)^2 [1+\alpha_6 \end{array}$
	$\begin{array}{c} \frac{4}{-[-1+\sum_1 (-\alpha_6)^u \alpha_{5-u}^{1+\pi(1+r_1 x} \\ \alpha_6)] \alpha_6^2 / \pi^2 [1+\alpha_6(r_1+\eta_1)]} \end{array}$	$\begin{array}{c} \frac{4}{[-1+\sum_1 (-\alpha_6)^u \alpha_{5-u}^{1+\pi(1+r_1 x} \\ \alpha_6)] \alpha_6 (1+\alpha_6 r) / \pi^2 [1+\alpha_6 \\ (r_1+\eta_1)]} \end{array}$	$\begin{array}{c} \frac{4}{-[-1+\sum_1 (-\alpha_6)^u \\ \alpha_6] [1+\alpha_6 r]^2 \\ +\eta_1]} \end{array}$

0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	$\frac{5}{0} \alpha_{q+1} r^q / (r-r_4-\eta_4)^2 [1 + \alpha_6 (r-\eta_5)]^2$	$\alpha_6 \frac{5}{0} \alpha_{q+1} r^q / [r-r_4-\eta_4] \times [1 + \alpha_6 (r-\eta_5)]^2$
0	$\frac{5}{1} \alpha_{q+1} r^{q-1} - C_1 [1 + \alpha_6 (r-\eta_5)] \times [r-r_4-\eta_4+1] / [r-r_4-\eta_4]^2 \times [1 + \alpha_6 (r-\eta_5)]$	$\alpha_6 [\frac{5}{1} \alpha_{q+1} r^{q-1} - \alpha_6 C_1 (r-r_4-\eta_4)] / (r-r_4-\eta_4) [1 + \alpha_6 (r-\eta_5)]^2$
0	$\frac{5}{2} \alpha_{q+1} r^{q-2} q(q-1) - 2C_1 \alpha_6 - 2C_2 \times [1 + \alpha_6 (r-\eta_5)] / 2 (r-r_4-\eta_4)^2 [1 + \alpha_6 (r-\eta_5)]$	$\alpha_6 [\frac{5}{2} \alpha_{q+1} r^{q-2} q(q-1) - 2\alpha_6 \times (C_2 (r-r_4-\eta_4) - 2C_1)] / 2 (r-r_4-\eta_4) [1 + \alpha_6 (r-\eta_5)]^2$
$r_4^{q-1} / \frac{3}{1} (r_4-r_i)$ $(r_4-\eta_5)]$	0	$\alpha_6 \frac{6}{1} \alpha_q r_4^{q-1} \frac{3}{0} \sum_0 (r_4-r)^v C_{v+1} / \frac{2}{i \neq 4} (r_4-r_i-\eta_i) [1 + \alpha_6 (r_4-\eta_5)]^2$
$\alpha_{5-u} + \pi(1+r_i \times$ $/\pi^2 [1 + \alpha_6 (r_i$	$[1 - \sum_1^4 (-\alpha_6)^u \alpha_{5-u} + \pi(1+r_i \alpha_6)] \times$ $[\sum_0^3 (-\alpha_6)^{3-v} (1 + \alpha_6 r)^v C_{v+1}] /$ $\pi^2 [1 + \alpha_6 (r_i + \eta_i)]$	0

$$E^{(j+1)} = M E^{(j+1)} + N E^{(j)} \Rightarrow E^{(j+1)} = (I-M)^{-1} N E^{(j)}.$$

Note that $I-M$ is a lower triangular matrix with diagonal elements 1 since M is subtriangular and hence $(I-M)^{-1}$ is also a triangular matrix with diagonal elements 1. Moreover, each element of $(I-M)^{-1}N$ consists of finite sum of terms each of which has for a factor an element of N . Therefore each element of $(I-M)^{-1}N$ can be made arbitrarily small in magnitude with the $|\alpha_q|$'s. Thus for sufficiently small magnitudes of α_q , $\max_w [\sum_u |p_{wu}|] < 1$ where $(I-M)^{-1}N = (p_{wu})$. Therefore the eigenvalues of (p_{wu}) are all less than 1. This proves the convergence of the iteration scheme established.

ILLUSTRATIVE EXAMPLES

A. As a Method for Determining the Zeros of a Polynomial from the Known Zeros of Its Lower-Degreed PartExample 1

Suppose the zeros of $h(x) = x^4 + 99.9999924x^3 - 800.4999542x^2 + 1400.9990082x - 1199.794521$ are to be determined wherein $99.9999924x^3 - 800.4999542x^2 + 1400.9990082x - 1199.794521$ had been previously studied and its zeros approximated to be 6 and $1 \pm i$. We may consider instead

$$g(x) = \frac{h(x)}{99.9999924} = 01x^4 + f(x)$$

where

$$f(x) = x^3 - 8.005001x^2 + 14.0099911x - 11.9979465,$$

and apply the iteration on $g(x)$.

$$\text{Here } \alpha_1 = .0020536$$

$$\alpha_2 = .0099909$$

$$\alpha_3 = -.0050001$$

$$\alpha_4 = .0$$

$$\alpha_5 = .01$$

the first three values, computed by the machine.

$$r_1\text{'s: } 1 \pm i$$

$$6.0$$

η_1	η_2	η_3	η_4
$-.0005385 +$	$.0026878i$	$-.4939229$	-7.5050001
$-.0005804 +$	$.0027260i$	$-.4349750$	-7.5683575
$-.0005656 +$	$.0026923i$	$-.4347241$	-7.5641376
$-.0005655 +$	$.0026923i$	$-.4347439$	-7.5641173

Therefore, the zeros of $g(x)$, and consequently of $h(x)$, are

$$\begin{array}{r} .9994345 + 1.0026923i \\ 5.5652561 \\ -107.5641174 \end{array}$$

as compared with the actual zeros of $h(x)$:

$$\begin{array}{r} .9994345 + 1.0026923i \\ 5.5652561 \\ -107.5641174. \end{array}$$

Example 2

$$\text{Let } h(x) = .001x^5 + x^4 - 4x^3 + 8x^2 - 8x + 4$$

Suppose the zeros of $x^4 - 4x^3 + 8x^2 - 8x + 4$ are previously known to be $1+i$, each one of multiplicity 2. To determine the zeros of $h(x)$:

$$r_i \text{'s: } \begin{array}{l} 1+i \\ 1+i \end{array}$$

$$\alpha_i \text{'s: } \begin{array}{l} 0. \\ 0. \\ 0. \\ 0. \\ 0.001 \end{array}$$

CO_{11}, CO_{12}	CO_{21}, CO_{22}	η_5
$.001 + .001i$	$.004 + .001i$	-3.9920000
$.000995 + .0009965i$	$.0039801 + .0009975i$	-3.9919533

Hence, the iteration converges after only two iterations, giving the zeros

$$r_i^* \text{'s: } \begin{array}{l} -1003.9919434 \\ 1.0123783 + 0.9648925i \\ .9836415 + 1.03411i \end{array}$$

as compared to the computed zeros [3],

$$\begin{array}{r} -1003.992040 \\ 1.0123775 + .96489419i \\ .9836425 + 1.0342299i. \end{array}$$

B. As Corrective Measure for Results Obtained
by Other Methods

Example 3

$$f(x) = x^4 - 2.075x^3 + .668x^2 - .091x + 2.28$$

(from Example 2 of Chapter II)

$$r_1 \text{'s: } \begin{array}{l} 1.50869 \pm .613861 \\ -.47119 \pm .797821 \end{array}$$

$$\alpha_1 \text{'s: } \begin{array}{l} .0023281 \\ .0005739 \\ .0000122 \\ .0 \\ .0 \\ .0 \end{array}$$

$$\begin{array}{ll} \eta_1, \eta_2 & \eta_3, \eta_4 \\ \begin{array}{l} .0001738 \pm .0001884i \\ .0001738 \pm .0001883i \end{array} & \begin{array}{l} -.0001738 \pm .0002940i \\ -.0001738 \pm .0002939i \end{array} \end{array}$$

After two iterations, the zeros are found to be

$$\begin{array}{l} 1.5088637 \pm .6140483i \\ -.4713638 \pm .7981139i. \end{array}$$

$$\text{Check: } \pi(x-r_1^*) = x^4 - 2.075x^3 + .6679998x^2 - .0909998x + 2.2800001.$$

Example 4

$$f(x) = x^4 + 200x^3 - 1600x^2 + 2800x - 2400$$

(from Flanagan-Maxfield's example [3])

$$r_1 \text{'s: } \begin{array}{l} .999615 + 1.001931i \\ 5.75077 \\ -208.0 \end{array}$$

$$\alpha_1 \text{'s: } \begin{array}{l} -3.9790344 \\ 3.4766235 \\ -1.5002289 \\ -.25 \\ 0. \\ 0. \end{array}$$

η_1, η_2	η_3	η_4
$-.0000096 \pm .0000051$	$.0161016$	$.2339176$
$-.0000096 \pm .0000051$	$.0161192$	$.2339000$

Therefore, the zeros are:

$$\begin{aligned} & .9996054 \pm 1.001925i \\ & 5.7668892 \\ & -207.7660999 \end{aligned}$$

as compared with actual zeros, [3]:

$$\begin{aligned} & .9996053 \pm 1.001925i \\ & 5.766889 \\ & -207.7660998. \end{aligned}$$

Example 5

$$f(x) = x^4 + 10.65x^3 + 129x^2 + 203.5x + 70$$

(from Lin's example in his 1941 iterative process [7])

$$\begin{aligned} \text{estimates } r_i \text{'s:} & \quad -1.3775 \\ & \quad -0.3775 \\ & \quad -4.455 \pm 9.65i \end{aligned}$$

$$\begin{aligned} \alpha_i \text{'s are: } & 11.2551432 \\ & 0.6052341 \\ & -0.1265793 \\ & -0.015 \\ & 0.0 \\ & 0.0 \end{aligned}$$

η_1	η_2	η_3, η_4
$.0996209$	$-.1003151$	$.0078471 \pm .0070405i$
$.1109277$	$-.1116175$	$.0078489 \pm .0070548i$
$.1123392$	$-.1130373$	$.0078490 \pm .0070557i$
$.1125191$	$-.1132171$	$.0078491 \pm .0070558i$
$.1125419$	$-.1132400$	$.0078491 \pm .0070558i$

Therefore, the zeros r_i^* are:

$$\begin{aligned} & -1.2649581 \\ & -0.4907400 \\ & -4.4471509 \pm 9.6429441i. \end{aligned}$$

$$\begin{aligned} \text{Check: } \pi(x-r_i^*) &= x^4 + 10.6499999x^3 + 128.9999943x^2 + 203.4999714x \\ & \quad + 69.9997603. \end{aligned}$$

Example 6

$$f(x) = x^4 - 6x^3 + 47x^2 - 18x + 290$$

(Lyon's example using Lin's 1941 method [7,10])

$$\text{estimates } r_i \text{'s:} \quad \begin{array}{r} -0.2855 + 2.545i \\ +3.2855 - 5.78i \end{array}$$

$$\text{the } \alpha_i \text{'s are:} \quad \begin{array}{r} .0936661 \\ -.1437273 \\ -.0094032 \\ -.0000001 \\ 0. \\ 0. \end{array}$$

$$\begin{array}{cc} \eta_1, \eta_2 & \eta_3, \eta_4 \\ \begin{array}{r} -.0010757 + .0021197i \\ -.0010762 + .0021207i \end{array} & \begin{array}{r} .0010758 + .0014573i \\ .0010762 + .0014580i \end{array} \end{array}$$

Therefore the zeros are

$$\begin{array}{r} 3.2844238 + 5.7778792i \\ -0.2844237 - 2.5464579i \end{array}$$

$$\text{Check: } \pi(x-r_i^*) = x^4 - 6.0000002x^3 + 46.9999995x^2 - 18.0000036x + 289.9999809.$$

Example 7

$$f(x) = x^6 + 256.79x^5 + 152.01x^4 + 258.43x^3 + 88.37x^2 + 4.52x + .0443 \quad (\text{Luke's example [9]})$$

$$r_i \text{'s:} \quad \begin{array}{r} -.0459 \\ -.0130 \\ -.107 + .9465i \\ -.316 \\ -256.2 \end{array}$$

$$\alpha_i \text{'s are:} \quad \begin{array}{r} .0004693 \\ .0042755 \\ .1043482 \\ .1564789 \\ .1270771 \\ .0011024 \\ 0. \\ 0. \end{array}$$

η_1	η_2	η_3, η_4	η_5	η_6
-.0006630	.0000385	-.001060 + .0001377i	.0003443	-.0006102
-.0006629	.0000378	-.001061 $\mp$.0001377i	.0003450	-.0006102

Therefore the zeros r_i^* 's are:

$$\begin{aligned}
 &-.465629 \\
 &-.0129622 \\
 &-.1071061 \pm .9466377i \\
 &-.3156550 \\
 &-256.2006035
 \end{aligned}$$

$$\begin{aligned}
 \text{Check: } \pi(x-r_1^*) &= x^6 + 256.7899933x^5 + 152.0099888x^4 + 258.4299774x^3 \\
 &+ 88.3699942x^2 + 4.5199998x + .0443000.
 \end{aligned}$$

Example 8

$$f(x) = x^4 - .94x^3 + .6x^2 + 2.99x + 10.45$$

(from Hitchcock's example using Lin's 1941 iteration [7,4])

$$\begin{aligned}
 r_i \text{'s are: } &-1.04 + 1.08i \\
 &1.51 \mp 1.55i
 \end{aligned}$$

$$\begin{aligned}
 \alpha_i \text{'s are: } &-.0764844 \\
 &.0391521 \\
 &-.0490000 \\
 &0.0 \\
 &0.0
 \end{aligned}$$

η_1, η_2	η_3, η_4
-.0028549 $\mp$.0087951i	.0028549 $\mp$.0049815i
-.0028488 $\mp$.0083380i	.0028489 $\mp$.0049921i

Therefore the zeros are:

$$\begin{aligned}
 &-1.0428488 + 1.0711662i \\
 &1.5128489 \mp 1.5450079i
 \end{aligned}$$

$$\begin{aligned}
 \text{Check: } \pi(x-r_1^*) &= x^4 - .9400002x^2 + .6000012x^2 + 2.989999x \\
 &+ 10.4500015.
 \end{aligned}$$

Example 9

$$f(x) = x^4 - 5.92x^3 + 13.661x^2 - 15.405x + 7.5308512$$

$$\text{estimated } r_1 \text{'s: } \frac{1+i}{2}, 2$$

$$\alpha_1 \text{'s are: } \begin{array}{l} -.4691488 \\ .5950000 \\ -.3390000 \\ -.0800000 \\ 0. \\ 0. \end{array}$$

η_1, η_2	CO_1	CO_2
$.0085372 + .019251i$	$.0024258$	$.0970745$
$.0099651 + .0199717i$	$.0024946$	$.0997774$
$.0099967 + .0199986i$	$.0025000$	$.0999938$
$.0100000 + .0200000i$	$.0025002$	$.0999996$

Therefore the zeros are:

$$1.0099999 \pm .9800000i \text{ as compared to the } 1.01 \pm .98i$$

$$1.9500002 \pm .0004553i \text{ actual zeros: } 1.95; 1.95.$$

Example 10

$$f(x) = x^6 + 256.8x^5 + 151.9x^4 + 258.5x^3 + 88.5x^2 + 4.5x + .05$$

(from Example 4 of Chapter II)

$$r_1 \text{'s are: } \begin{array}{l} -.484625 + 1.09541i \\ -.0157900 \\ -.0428000 \\ .4486598 \\ -256.2111402 \end{array}$$

$$\alpha_1 \text{'s are: } \begin{array}{l} .1614605 \\ 13.9904507 \\ 238.3653011 \\ -5.0884171 \\ 2.4765129 \\ 0.0096817 \\ 0. \\ 0. \end{array}$$

η_1, η_2	η_3	η_4	η_5	η_6	
.2178881 + .3283407 +	.2009548i .2092154i	.0000004 .0000152	.0000583 .0010245	-.4455926 -.7035223	.0000761 .0000761
.3809192 + .3872277 +	.1660623i .1384078i	.0000013 .0000008	-.0002359 -.0001477	-.8032323 -.7891522	.0000761 .0000761
.3795185 + .3755483 +	.1413487i .14970398	-.0000008 -.0000009	-.0001461 -.0001628	-.7598841 -.7581017	.0000761 .0000761
.3770083 + .3783201 +	.1504719i .1485263i	-.0000009 -.0000009	-.0001667 -.0001627	-.7652827 -.7672360	.0000761 .0000761
.3782191 + .3778706 +	.1492440i .1482930i	-.0000009 -.0000009	-.0001609 -.0001615	-.7658265 -.7650339	.0000761 .0000761
.3778220 + .3779017 +	.1485242i .1482930i	-.0000009 -.0000009	-.0001621 -.0001620	-.7652305 -.7654734	.0000761 .0000761
.3779304 +	.1484091i	-.0000009	-.0001619	-.7654734	.0000761

Therefore the zeros r_i^* are:

$$\begin{aligned}
 &-.1066945 + .9469909i \\
 &-.0157909 \\
 &-.0429619 \\
 &-.3168139 \\
 &-256.2110634.
 \end{aligned}$$

Check: $\pi(x-r_i^*) = x^6 + 256.8000183x^5 + 151.9045715x^4 + 258.5060730x^3$
 $+ 88.5159674x^2 + 4.5009227x + .0500106.$

Example 11

$$f(x) = x^6 - 12x^5 + 50x^4 - 70x^3 - 41x^2 + 2x + 390$$

(from Example 5 of Chapter II)

r_i 's are: $-1.1068842 + 1.2331$
 $2.9771064 + 2.09291$
 3.2595779
 4.9999778

α_i 's are: -202.5803146
 90.9204035
 -15.9860919
 1.1802006
 0.0001483
 0.0
 0.0
 0.0

η_1, η_2	η_3, η_4	η_5	η_6
.0959058 $\mp$.1959683i	.0141544 $\mp$.086504i	-.2201437	.0000232
.1051948 $\mp$.2274607i	.0216165 $\mp$.0923846i	-.2554066	.0000226
.1067177 $\mp$.2322430i	.0227807 $\mp$.0928167i	-.2590590	.0000224
.1068525 $\mp$.2328959i	.0228784 $\mp$.0928907i	-.2597132	.0000224
.1068811 $\mp$.2329856i	.0228919 $\mp$.0928983i	-.2595686	.0000224

Therefore the zeros r_i^* 's are:

$$\begin{aligned} & -1.0000031 + 1.0000144i \\ & 2.9999982 + 2.0000171i \\ & 3.0000092 \\ & 5.0000002 \end{aligned}$$

as compared to the actual zeros:

$$-1 \pm i; \quad 3 \pm 2i; \quad 3; \quad 5.$$

C. As a Method, Independent of Other Methods,
for Finding the Zeros of a Polynomial

Example 12

$$f(x) = x^4 - 5.0000002x^3 + 3.0000029x^2 - 2.0000224x + 1.0000006$$

Noting that $f(0) > 0$ and $f(1) < 0$ and that $f(4) < 0$ while $f(5) > 0$, so that there are real zeros between 0 and 1 and between 4 and 5, we take for initial guesses .5 and 4.5 for the real zeros of $f(x)$, and since $f(x)/(x-.5)(x-4.5) \approx x^2 + .75$ whose zeros are $\approx \pm .87i$, we take $\pm .87i$ as the remaining values of r_i 's.

Here α_i 's are:

$$\begin{aligned} & -.7030244 \\ & 1.7844775 \\ & -.0068971 \\ & -.0000002 \\ & .0 \\ & .0 \end{aligned}$$

η_1	η_2	η_3, η_4
.0465513	-.0855367	.0194928 $\mp$.2118031i
.0721217	-.0886485	.0104368 $\mp$.2406905i
.0754045	-.0890344	.0069015 $\mp$.2423776i
.0752518	-.0889806	.0068349 $\mp$.2422557i
.0752249	-.0889739	.0068747 $\mp$.2422516i

Therefore, the zeros are: .5752249
 4.4110261
 .0068747 $\pm$.6277484i

as compared with the actual zeros

.5752290
 4.4110253
 .0068730 $\pm$.6277460i.

Example 13

$$h(x) = x^3 + 8.0000038x^2 + 8.0799952x - 15.8399957$$

Considering $x^3/8.0000038 + (x^2 + 1.0099990x - 1.9799985)$,

the zeros of the quadratic part are readily obtainable, in fact,
 ≈ -2 and 1 since the quadratic polynomial $\approx x^2 + x - 2$. So temporarily truncating the cubic polynomial, we get for our input

r_1 's: -2 and 1

α_i 's: .0200015
 .0099990
 .0
 .1249999

η_1	η_2	η_3
-.3333319	-.0516668	1.3849987
-.5879487	-.0488511	1.5871481
-.6142955	-.0466220	1.6612947
-.6243220	-.0467498	1.6690061
-.6254606	-.0466695	1.6721429
-.6258970	-.0466748	1.6724822
-.6259472	-.0466714	1.6726190
-.6259662	-.0466716	1.6726339

Therefore the zeros of $h(x)$ are: -2.6259662
 $.9533284$
 -6.3273725

as compared with the actual zeros: -2.6259698
 0.9533286
 $-6.3273627.$

Example 14

$$h(x) = x^5 + 9.9999998x^4 + .0000015x^3 - 9.9999975x^2 \\ + .0000048x - 19.9999990$$

The zeros of $h(x)$ are the same as those of $g(x) = .1x^5 + f(x)$
 where $f(x) = x^4 + .0000002x^3 - .9999998x^2 + .0000005x - 1.9999999$
 $\approx x^4 - x^2 - 2$

Since the zeros of $x^4 - x^2 - 2$ are readily found, we use the technique
 of temporarily truncating $g(x)$ to obtain the zeros $\pm \sqrt{2}$ and $\pm i$ of $f(x)$.

So take r_1 's: $\pm i$
 ± 1.41421

α_1 's: $-.0000100$
 $.0000005$
 $-.0000099$
 $.0000002$
 $.0$
 $.1$

η_1, η_2	η_3	η_4	η_5
.0166668	-0.0666629	-0.0666639	.0999992
.0167226 $\pm$.0010804i	- .0584762	-.0792109	.1041242
.0165776 $\pm$.0010555i	-.0582909	-.0790671	.1042036
.0165772 $\pm$.0010561i	- .0582852	-.0790720	.1042031

Therefore, the zeros are: $.0165772 \pm .9989439i$
 1.3559247
 -1.4932820
 -9.8957968

as compared with the actual zeros:

$.0165771 \pm .9989439i$
 1.3559249
 -1.4932819
 $-9.8957971.$

Conclusions

We have thus established that the iterative formulas proposed in this paper converge whenever the $|\alpha_q|$'s in

$$\alpha_{n+2} x^{n+1} + \sum_{i=1}^{n+1} (a_i + \alpha_i) x^{i-1}$$

are sufficiently small. As a means of solving for the zeros of

$$\sum_{i=1}^{n+1} (a_i + \alpha_i) x^{i-1},$$

that is, $\alpha_{n+2} = 0$, that condition amounts to requiring that the estimates r_i 's of the actual zeros r_i^* 's, which are input of the program, be not too far off. However, as shown in Examples 10 and 11, the process frequently converges even when the α_q 's are as big as a three-digit figure.

The iteration is then particularly effective as a corrective measure to other methods which obtain the zeros one at a time and thus are "contaminating" procedures. Note from the example that as such, this noncontaminating iterative process usually obtains the actual zeros after only a few iterations.

As a method, by itself, of obtaining the zeros of a polynomial, it is especially applicable to a polynomial of degree 3 or one of higher degree, wherein ignoring the leading term gives a polynomial of lower degree whose zeros are readily obtainable or are known. (Examples 13 and 14 illustrate this point.) Then, after the zeros of the truncated polynomial are found, the zeros of the original polynomial can be

determined. Here, $\alpha_{n+2} \neq 0$ and the iterative process seems to converge when $|\alpha_{n+2}|$ is quite small, that is, when the second coefficient of the original monic polynomial is rather large in magnitude.

APPENDICES

```

C      APPENDIX A. MACHINE PROGRAMS IN FORTRAN II FOR IBM 709
      LIN'S PENULTIMATE REMAINDERING WITH TRANSLATION FOR QU
      ADRATIC OR L
      LINEAR FACTOR, WITH GCD ALGO TO DETERMINE PRESENCE OF MU
      LTIPLE ZERCS
      DIMENSION A(20), A1(20), B(2,20), C(20), D(20), AD(20)
I      DIMENSION G(10), Y(2)
      1 READINPUTTAPE5, 10, N, JLIM
      10 FORMAT(2I2)
      READINPUTTAPE5, 14, (A(I), I=1,N), H
      14 FORMAT(4F15.7)
C      GCD ALGORITHM TO DETERMINE PRESENCE OF MULTIPLE ZERCS
      DO 49 I=1,N
      49 A1(I)= A(I)
      N1 = N-1
      AN= N
      DO 51 I=1,N1
      A1=I
      51 AD(I)= A1*A1(I+1)/AN
      NN=N
      M=N1
C      PERFORM DIVISION OF A'S BY AD'S
      79 N1= NN-1
      DO 53 I=2,M
      MN= NN-M+I
      53 C(MN)= 0.0
      MN=NN-M+1
      C(MN)= 1.0
      DO 55 I=M,N1
      DELTA = 0.0
      DO 57 K=1,M
      MN= NN+M+1-I-K
      57 DELTA= C(MN)*AD(K) + DELTA
      NI= NN-I
      NIM= NI+M
      55 C(NI)= A1(NIM)- DELTA
      NM1= NN-M+1
      C(NM1)= 1.0
      DO 59 I=1,M
      DELTA =0.0
      DO 61 K=1,I
      IK= I+1-K
      61 DELTA= DELTA+ AD(K)*C(IK)
      59 D(I) = A1(I)- DELTA
C      CHECK WHETHER DIVISION NEED BE REPEATED OR GCD IS OBTAINED
      DO 63 I=1,M
      MM= M-I+1
      IF(ABSF(D(MM))-1.0E-5) 63,63,65
      63 CONTINUE
      WRITEOUTPUTTAPE6, 67, (AD(I), I=1,M)
      67 FORMAT (14H THE GCD IS / (6F12.6))
      GO TO 81
      65 IF(MM-1) 69,69,71

```

```

69 WRITEOUTPUTTAPE6, 73
73 FORMAT(9H GCD IS 1)
   GO TO 91
71 NN=M
   M= M-1
   DO 75 I=1,NN
75 A1(I)= AD(I)
   DO 77 I=1,M
77 AD(I)= D(I)/D(MM)
   GO TO 79
81 DO 83 I=2,M
   MN= N-M+1
83 C(MN)= 0.0
   MN= N-M+1
   C(MN)= 1.0
   NI= N-1
   DO 85 I=M,NI
   DELTA= 0.0
   DO 97 K=1,M
   MN= N+M+1-I-K
97 DELTA= C(MN)*AD(K) + DELTA
   NI= N-I
   NIM=NI+M
85 C(NI)= A(NIM)-DELTA
   N=N-M
   DO 89 I=1,N
89 A(I)= C(I)
91 WRITEOUTPUTTAPE6, 32, (A(I), I=1,N)
32 FORMAT(46H THE COEFF OF POLYNOMIAL BEING FACTORED ARE
          /(4F15.7))
C   IF(N-2) 1,1,93
   PERFORM TRANSLATION ON X OF F(X)
93 DO 174 I=1,N
174 C(I) = A(I)
   DO 175 I= 1,N
   ND = N+2-I
   D(ND) = 1.0
   DO 176 K=1,N
   NK= N+1-K
176 D(NK) = C(NK) + H*D(NK+1)
   A1(I)= D(1)
   IF(I-N) 200, 175,175
200 NI= N-I
   DO 201 K=1,NI
201 C(K) = D(K+1)
175 CONTINUE
   A1(N+1)= 1.0
   WRITEOUTPUTTAPE6, 32, (A1(I), I=1,N)
C   FORM THE 1ST TRIAL DIVISOR
   IF(XMODF(N,2)) 22,23,22
22 M = 1
   GO TO 25
23 M=2
25 IF(ABSF(A1(M+1))-1.0E-5) 27,27,29

```

```

27 WRITEOUTPUTTAPE6, 20
20 FORMAT(39H C DIVISOR, CHANGE STARTING VALUES OF B)
GO TO 1
29 DO 31 I=1,M
31 B(I,1)= A1(I)/A1(M+1)
C   PERFORM DIVISION
    J=1
34 WRITEOUTPUTTAPE6, 16, J
16 FORMAT(1X, 18H ITERATION NO. 1S ,I2)
   WRITEOUTPUTTAPE6, 14, (B(I,J), I=1,M)
   DO 36 I=2,M
     MN = N-M+1
36 C(MN)= 0.0
     MN= N-M+1
     C(MN)= 1.0
     N1=N-1
     DO 38 I=M,N1
       DELTA = 0.0
       DO 40 K=1,M
         MN= N+M+1-I-K
40 DELTA = C(MN)*B(K,J)+DELTA
       N1 = N-I
       NIM = N-I+M
38 C(NI)=A1(NIM)-DELTA
       IF(J-1) 42,42,44
44 T=ABSF(B(1,J)-B(1,J-1))
       IF(T-1.0E-3) 50,50,48
50 IF(M-1) 1,68,46
46 T= ABSF(B(2,J)-B(2,J-1))
       IF(T-1.0E-3) 70,70,48
48 IF(J-JLIM) 42,42,54
42 IF(ABSF(C(1))-1.0E-5) 27,27,56
56 B(1,J+1) = A1(1)/C(1)
       IF(M-1) 1, 62, 64
64 B(2,J+1)= (A1(2)-C(2)*B(1,J))/C(1)
62 J=J+1
    GO TO 34
54 WRITEOUTPUTTAPE6, 66,N
66 FORMAT(40H ITERATION NOT CONVERGING FOR DEGREE /12)
GO TO 1
C   OBTAIN THE ZEROS
68 X= -B(1,J)+ H
   WRITEOUTPUTTAPE6, 106, X
106 FORMAT (13H ZEROS ARE /((2F12.5))
GO TO 72
70 DO 74 I=1,2
I   G(I) = (0.0,C.0)
74 G(I) = B(I,J)
I   82 CALL QUADRA(G,Y)
I   HH = (0.0,0.0)
     HH = H
     DO 87 I=1,2
I   87 Y(I) = Y(I) + HH
     WRITEOUTPUTTAPE6, 106, Y(1), Y(3), Y(2), Y(4)

```

```

72 N=N-M
   IF(N) 1,1,84
84 DO 80 I=1,N
80 A(I) = C(I)
   IF(N-2) 1, 150, 12
150 M=2
   DO 151 I=1,2
I   G(I)= (0.0,0.0)
151 G(I)= A(I)
   GO TO 82
12 WRITE(OUTPUT,TAPE6, 32, (A(I), I=1,N)
   GO TO 23
END
SUBROUTINE QUADRA(G,Y)
I   DIMENSION G(10), Y(2)
I   DELTA= SQRTF(G(2)**2-(4.0,0.0)*G(1))
I   Y(1) = (-G(2)+DELTA)/(2.0,0.0)
I   Y(2)= (-G(2)-DELTA)/(2.0,0.0)
I   RETURN
END

```

NDFILE

For the data, n = the degree of the polynomial $f(x)$ (≤ 20)

$jlim$ = upperbound for the iteration number

h = amount of translation on x

$\{A(i)\}$ = real coefficients of $f(x) = \sum_{i=1}^{n+1} A(i)x^{i-1}$

If $GCD = 1$, Penultimate Remaindering is applied to $f(x+h)$.

If GCD is a polynomial of positive degree, Penultimate Remaindering is applied to f/GCD , in order to have simple zeros. The multiplicity of the zeros found can later be determined either by studying the GCD or by repeatedly depressing f by the zeros obtained.

```

C      A NONCONTAMINATING ITERATIVE PROCESS FOR DETERMINING T
      HE ZEROS OF
1A POLYNOMIAL
   DIMENSION NT(5)
I      DIMENSION A(10), ALPHA(12), C(10), CC(10,10,20), D(10)
      , ETA(11,20)
      1,G(10), H(10, 5), R(10), RR(11), X(11), Y(2), PNUM(10)
      , PNUM1(10,5
      2), DENOM(10), SUM(5,10), SSUM(5)
100 READINPUTTAPE5, 1, N, NS,L
      1 FORMAT (3I3)
      N1 = N+1
      N2 = N+2
      READINPUTTAPE5, 2, (A(I), A(I+10), I=1,N)
      READINPUTTAPE5, 2, (R(I), R(I+10), I=1,N)
      READINPUTTAPE5, 2, ALPHA(N2), ALPHA(N2+12)
      WRITEOUTPUTTAPE6, 7
      7 FORMAT(46H HERE ARE THE A'S, R'S + ALPHA'S, RESPECTIVE
      LY)
      WRITEOUTPUTTAPE6, 2, (A(I), A(I+10), I=1,N)
      WRITEOUTPUTTAPE6, 2, (R(I), R(I+10), I=1,N)
      2 FORMAT(4F15.7)
      IF(L) 3,3,4
      4 READINPUTTAPE5, 1, (NT(I), I=1,L)
C      OBTAIN ALPHA'S
I      3 C(1) = -R(1)
      DO 10 I=2,N
I      C(1) = -R(I)
I      CALL MULTIP(C,D,G,I-1,1)
      DO 10 J=1,I
I      10 C(J) = G(J)
      DO 11 J=1,N
I      11 ALPHA(J) = A(J)-G(J)
I      ALPHA(N1) = (0.0,0.0)
      WRITEOUTPUTTAPE6, 2, (ALPHA(J), ALPHA(J+12), J=1,N2)
C      FOR SIMPLE R'S AND R(N+1)
      J=1
I      ETA(N1,2) = G(N) - ALPHA(N2)*(G(N-1)-G(N)**2) + ALPHA(
      N)
      IF(NS) 27,27,12
      12 DO 13 K=1,NS
I      PNUM(K) = (0.0,0.0)
I      DENOM(K) = (1.0,0.0)
      DO 14 I= 1,N2
I      14 PNUM(K) = PNUM(K)- ALPHA(I)*R(K)**(I-1)
      DO 15 I=1,N
      IF (K-I)16,15,16
I      16 DENOM(K) = DENOM(K)*(R(K)-R(I))
      15 CONTINUE
I      13 ETA(K,J+1) = PNUM(K)/DENOM(K)
      GO TO 17
I      18 IF(ABSF(ALPHA(N2))) 19,19,20
I      20 PNUM2 = (0.0,0.0)
I      PNUM3 = (1.0,0.0)

```

```

      DO 21 I= 1,N
      IN = N-I+1
I      PNUM2 = PNUM2 + ALPHA(IN)*(-ALPHA(N2))*I
I 21  PNUM3 = ((1.0,C.0) + R(I)*ALPHA(N2))*PNUM3
I      PNUM4 = (1.0,0.0) - PNUM2-PNUM3
I      DENOM1 = ALPHA(N2)
      IF(NS) 22,22,23
23  DO 24 I=1,NS
I 24  DENOM1 = DENOM1 *((1.0,.0)+ALPHA(N2)*(R(I)+ETA(I,J)))
      IF (L) 25,25,22
22  DO 26 I=1,L
I 26  DENOM1 = DENOM1*SSUM(I)
I 25  ETA(N1,J+1)= PNUM4/DENOM1
19  IF(NS) 27,27,28
28  DO 29 K = 1,NS
I      DENOM(K) = (1.0,0.0)
      DO 30 I=1,NS
      IF(K-I) 31,30,31
I 31  DENOM(K)= DENOM(K)*(R(K)-R(I)-ETA(I,J))
30  CONTINUE
      IF(L) 32,32,33
33  DO 34 I=1,L
I 34  DENOM(K) = DENOM(K)*SUM(I,K)
I 32  IF(ABSF(ALPHA(N2))) 29,29,35
I 35  DENOM(K) = DENOM(K) *((1.0,0.0)+ALPHA(N2)*(R(K)-ETA(N1
      ,J)))
I 29  ETA(K,J+1) = PNUM(K)/DENOM(K)
17  WRITEOUTPUTTAPE6, 2, (ETA(K,J+1), ETA(K,J+21), K=1,NS)
I 27  IF(ABSF(ALPHA(N2))) 36,36,37
37  WRITEOUTPUTTAPE6, 2, ETA(N1,J+1), ETA(N1,J+21)
36  WRITEOUTPUTTAPE6, 38
38  FORMAT (32H ABOVE ARE ETAS FOR SIMPLE ZERUS)
C      FOR MULTIPLE ZEROS
      IF(L) 39,39,41
39  IF(J-1) 94,94,40
41  NSS = NS+1
      NS1 = NS+1
      DO 42 I=1,L
      NTI = NI(I)
      NSI = NSS
I      C(1) = (1.0,0.0)
      IF(NS) 43,43,44
44  DO 45 K=1,NS
      IF(J-1) 220,220,47
I 220 D(1)= -R(K)
      GO TO 46
I 47 D(1)= -R(K) - ETA(K,J)
I 46 CALL MULTIP(C,D,G,K-1,1)
      DO 48 KK=1,K
I 48 C(KK)= G(KK)
45  CONTINUE
43  NDEG= NS
      IF(J-1) 49,49,50
49  DO 51 K=NS1,N

```

```

      IF(R(K)-R(NSI)) 52,53,52
53 IF(R(K+10)-R(NSI+10)) 52,51,52
I 52 D(1) = -R(K)
I CALL MULTIP(C,D,G,NDEG,1)
    NDEG= NDEG+1
    DO 54 KK=1,NDEG
I 54 C(KK)= G(KK)
51 CONTINUE
    DO 61 K= 1,NTI
I PNUM1(K,I) = (0.0,0.0)
    K1 = K-1
    DO 61 M= K,N2
    M1 = M-1
I 61 PNUM1(K,I) = PNUM1(K,I)+PROD(M1,K1)*ALPHA(M)*R(NSI)**(
                                M-K)
    GO TO 55
50 DO 56 II=1,L
    NTII = NT(II)
    IF(I-II) 57,56,57
57 DO 58 K=1,NTII
I 58 D(K)= H(K,II)
I CALL MULTIP(C,D,G,NDEG,NTII)
    NDEG = NDEG + NTII
    DO 59 K = 1,NDEG
I 59 C(K) = G(K)
56 CONTINUE
I IF(ABS(F(ALPHA(N2)))) 210,210,67
I 67 CON= (1.0,0.0) + ALPHA(N2)*(R(NSI)-ETA(N1,J))
    GO TO 55
I 210 CON = (1.0,0.0)
I 55 RT = R(NSI)
I CALL DERIV(0,C,X,N-NTI, RT)
    DO 60 K= 1,NTI
    K1 = K-1
I DENOM2 = X(1)*PROD(K1,K1)
I PNUM2 = (0.0,0.0)
I PNUM3 = (0.0,0.0)
    IF(K1) 9,9,63
9 IF(J-1) 60,60,8
63 DO 64 M=1,K1
    K1M= K1-M
    KM= K-M
I RT= R(NSI)
I CALL DERIV(M,C,X,N-NTI, RT)
I 64 PNUM3 = PNUM3-PROD(K1,K1M)*X(M+1)*CO(KM,I,J+1)
    IF(J-1) 60,60,65
65 DO 66 M=1,K1
    K1M = K1-M
    KM= K-M
I 66 PNUM2 = PNUM2 + PROD(K-2, K1M)*CO(KM,I,J+1)*X(M)
I AK= K1
I PNUM2 = -ALPHA(N2)*AK*PNUM2
I PNUM3 = CON*PNUM3
I 8 DENOM2 = CON *DENOM2

```

```

I 60 CO(K,I,J+1) = (PNUM1(K,I) + PNUM2 + PNUM3) / DENOM2
      WRITEOUTPUTTAPE6, 2, (CO(K,I,J+1), CO(K,I,J+2)), K=1, NT
      I)
      WRITECPUTTAPE6, 69, I
69 FORMAT(35H ABOVE ARE C'S FOR MULTIPLE ZERO /13)
C DATA NEEDED FOR NEXT ITERATION OF BOTH THE SIMPLE AND
  MULTIPLE ZERO
  IOS
    IF(NS) 62,62,70
  70 DO 71 K=1,NS
I SUM(I,K) = (0.0,0.0)
    DO 271 KL=1,NTI
I 271 SUM(I,K) = SUM(I,K) + CO(KL,I,J+1) * (R(K) - R(NSI)) ** (KL-1
      )
I 71 SUM(I,K) = SUM(I,K) + (R(K) - R(NSI)) ** NTI
I 62 IF(ABSF(ALPHA(N2))) 68,68,72
I 72 SSUM(I) = (0.0,0.0)
    DO 73 K=1,NTI
I 73 SSUM(I) = SSUM(I) + CO(K,I,J+1) * ((-ALPHA(N2)) ** (NTI-K+1
      )) * ((1.0,0.0
      1) + ALPHA(N2) * R(NSI)) ** (K-1)
I SSUM(I) = SSUM(I) + ((1.0,0.0) + ALPHA(N2) * R(NSI)) ** NTI
68 DO 74 K=1,NTI
I 74 C(K) = CO(K,I,J+1)
    DO 75 M=1,NTI
      ND = NTI + 2 - M
I D(ND) = (1.0,0.0)
    DO 76 K=M,NTI
      NTK = NTI + 1 - K
I 76 D(NTK) = C(NTK) - R(NSI) * D(NTK+1)
I H(M,I) = D(1)
    IF (M-NTI) 200,75,75
200 NTM = NTI - M
    DO 201 K=1,NTM
I 201 C(K) = D(K+1)
    75 CONTINUE
    42 NSS = NSS + NTI
C TEST FOR CONVERGENCE
    IF(J-1) 94,94,78
  78 DO 80 I=1,L
    NTI = NT(I)
    DO 80 K=1,NTI
I T = ABSF(CO(K,I,J+1) - CO(K,I,J))
    IF(T-1.0E-4) 80,80,82
  80 CONTINUE
    IF(NS) 81,81,40
  40 DO 83 K=1,NS
I T = ABSF(ETA(K,J+1) - ETA(K,J))
    IF(T-1.0E-4) 83,83,82
  83 CONTINUE
I 81 IF(ABSF(ALPHA(N2))) 84,84,77
I 77 T = ABSF(ETA(N1,J+1) - ETA(N1,J))
    IF(T-1.0E-4) 84,84,82
  84 WRITEOUTPUTTAPE6, 85

```

```

85 FORMAT( 14H THE ROOTS ARE)
   IF(NS) 86,86,87
87 DO 88 K=1,NS
I   RR(K)= R(K)+ ETA(K,J+1)
88 WRITEOUTPUTTAPE6, 2, RR(K), RR(K+11)
I 86 IF(ABSF(ALPHA(N2)))97,97,99
I 99 RR(N1) = -(1.0,0.0)/ALPHA(N2) + ETA(N1,J+1)
   WRITEOUTPUTTAPE6, 2, RR(N1), RR(N1+11)
97 IF(L) 100,100,89
89 NSS= NS+1
   CO 90 I= 1,L
   NSI = NSS
   IF(NT(I)-2) 91,91,90
91 CO 92 K=1,2
I 92 G(K) = CO(K,I,J+1)
I   CALL QUADRA(G,Y)
   DO 93 K= 1,2
I   RR(K)= R(NSI)+Y(K)
93 WRITEOUTPUTTAPE6, 2, RR(K), RR(K+11)
90 NSS = NSS+NT(I)
   GO TO 100
82 IF(J-15) 94,94,95
94 WRITEOUTPUTTAPE6,96,J
96 FORMAT (25H STAGE OF ITERATION IS //I3)
79 J = J+1
   GO TO 18
95 WRITEOUTPUTTAPE6, 98
98 FORMAT( 25H THE ITERATION IS NOT CONVERGING)
   GO TO 100
   ENC
   SUBROUTINE MULTIP(C,D,G,NP,MP)
I   DIMENSION C(10), D(10), G(10)
   NP2 = NP+2
   MP2 = MP+2
   NM= NP+MP
   DO 101 K=NP2,NM
I 101 C(K)= (0.0,0.0)
   DO 102 K = MP2, NM
I 102 D(K)= (0.0,0.0)
   C(NP+1) = (1.0,0.0)
I   D(MP+1) = (1.0,0.0)
   DO 103 I= 1,NM
I   G(I) = (C.0,0.0)
   DO 103 J= 1,I
   IJ= I-J+1
I 103 G(I)= G(I)+C(J)*D(IJ)
I   RETURN
   END
   FUNCTION PRGD(NU,NV)
   MPRGD = 1
   IF(NU) 105,105,106
106 IF(NV) 105,105,107
107 DO 108 I= 1,NV
   NUI = NU-I+1

```

```

108 MPRCD = MPRCD*NU I
I 105 PRCD = MPRCD
I   RETURN
   END
   SUBROUTINE DERIV(M,G,X,ME,RI)
I   DIMENSION X(11), G(10)
I   X(M+1) = (0.0,0.0)
   IF(M-ME) 110,110,111
110 M1 = M+1
   GO 112 I=M1, ME
I 112 X(M1) = X(M1)+ PRUD(I-1,M)*G(I)*RT**(I-M1)
I   X(M1) = X(M1) + PROD(ME, M)*RT**(ME-M)
I 111 RETURN
   END
   SUBROUTINE QUADRA(G,Y)
I   DIMENSION G(10), Y(2)
I   DELTA = SQRTF(G(2)**2-(4.0,0.0)*G(1))
I   Y(1) = (-G(2)+DELTA)/(2.0,0.0)
I   Y(2) = (-G(2)-DELTA)/(2.0,0.0)
I   RETURN
   END

```

NDFILE

For the data, n = degree of the polynomial $f(x)$ (≤ 10)

ns = number of simple zeros of $t(x)$

L = number of multiple zeros of $t(x)$

$\{A(i)\}$ = coefficients of $f(x) = \sum_{i=0}^{n+1} A(i)x^{i-1}$

$\{r(i)\}$ = zeros of $t(x)$ with the $\overset{1}{\text{simple}}$ zeros listed first

and the multiple zeros repeated according to

respective multiplicities

$\{nt(i)\}$ = multiplicities of the multiple zeros of $t(x)$

$\alpha(n_2)$ = coefficient of x^{n+1} term of $g(x)$.

APPENDIX B

COMMON METHODS FOR DETERMINING ZEROS OF A POLYNOMIAL

I. For Real Zeros

1. Half Interval Search [6]

- (a) First approximate by graphical or any other means an interval containing a real zero of $f(x)$, say (d, b) . Assume $f(d) < 0$, for the purpose of discussion.
- (b) Test whether $f(\frac{d+b}{2})$ is less than or greater than or equal to 0. If $f(\frac{d+b}{2}) = 0$, then the required zero is obtained; if it is less than 0, then the required zero is in the interval $(\frac{d+b}{2}, b)$, whereas if greater than 0, it is in the interval $(d, \frac{d+b}{2})$.
- (c) So if still necessary, repeat the process with the interval (d, b) replaced by either $(\frac{d+b}{2}, b)$ or $(d, \frac{d+b}{2})$, as the case may be.
- (d) By continuing the process, we thus narrow down the interval as much as we want to, and hence locate a zero to an accuracy of any reasonable number of significant digits.

2. Method of False Position [6]

- (a) As in the previous method, start with an interval (d, b) in which $f(x)$ changes sign.

- (b) Linear interpolation is then used to find an approximate zero x , as follows:

Suppose $f(d) < 0$ and $f(b) > 0$. The line joining $f(d)$ and $f(b)$ intersects the real axis in x , where x is determined by the relation $\frac{-f(d)}{f(b) - f(d)} = \frac{x-d}{b-d}$, that is, by

$$x = d + \frac{f(d)(b-d)}{f(d)-f(b)}. \quad (\text{The same formula is derived when } f(d) > 0 \text{ and } f(b) < 0.)$$

- (c) To determine which of the two intervals (d,x) or (x,b) contains the true zero x_t , check the sign of $f(x)$:
 if $f(x) > 0$, then x_t is in (d,x) ; if < 0 , then x_t is in (x,b) ; if $= 0$, then $x_t = x$.
- (d) If the zero is not yet determined, repeat the procedure with (d,b) replaced by either (d,x) or (x,b) , as the case may be.
- (e) Continue the process until either $|f(x)| < \text{eps}$ or the length of the interval $< \text{eps}$, where eps is a predetermined small positive number.

3. Newton-Raphson Method [6]

- (a) Start with an arbitrary approximate to the zero, say x_1 .
- (b) To improve the approximation, draw the tangent to the curve $f(x)$, and determine its intersection with the real axis, say x_2 . The relation therefore is

$$\tan \alpha = f'(x_1) = \frac{f(x_1)}{x_1 - x_2}, \quad \text{that is, } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

- (c) The iterative sequence can now be set up as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

provided $f'(x_n) \neq 0$.

- (d) Repeat the iteration until $|f(x_{n+1})| < \text{eps}$, or when $|x_{n+1} - x_n| < \text{eps}$, or else when it shows signs of divergence such as when $|f'(x_{n+1})| < \text{eps}$ or the number of iterations exceeds jlim. Here jlim and eps are predetermined values.

II. For Real and Imaginary Zeros

1. Penultimate Remaindering [1,2]

- (a) Start with a first trial divisor to $f(x)$, usually the last m terms of $f(x)$, reduced to a monic polynomial divisor, where m is the degree of the divisor being sought.
- (b) Divide $f(x)$ by the approximate divisor, until a penultimate remainder is obtained, that is, a remainder of degree m .
- (c) Taking this penultimate remainder divided by its leading coefficient as the next trial divisor, the procedure is repeated.
- (d) The process is continued until the number of iterations exceeds a certain number without showing sign of

convergence, or else when the coefficients of the succeeding trial divisors converge.

A slight variation to this iteration is to take alternatively for trial divisor the penultimate remainder of degree $m+1$ and the ultimate remainder which is degree m , each time reduced to monic polynomial.

2. Lin's Ultimate Remaindering Iteration [7,8]

When $f(x) = \sum_1^{n+1} a_i x^{i-1}$ is divided by $d(x) = \sum_1^{m+1} b_i x^{i-1}$, the ultimate remainder has coefficients which are expressions involving the coefficients a 's, b 's, and the quotient coefficients c 's. If $d(x)$ were a factor of $f(x)$, each of those remainder terms is identically 0. So to obtain an iterative set of formulas for the b 's which hopefully will converge to the exact coefficients of $d(x)$, we set each of the terms in the expression for the remainder to 0, and solve for the b 's as functions of the a 's and the c 's.

To illustrate for the case when $m=2$, the quotient when $f(x)$ is divided by the quadratic $d(x)$ has coefficients expressed as a recursive relation $c_{n-k-1} = a_{n-k+1} - b_2 c_{n-k} - b_1 c_{n-k+1}$, $k = 1, 2, \dots, n-2$ where $c_n = 0$ and $c_{n-1} = 1$; while the remainder coefficients are $r = a_2 - b_2 c_1 - b_1 c_2$

$$s = a_1 - b_1 c_1.$$

Solving for b_1 and b_2 , after setting r and s to 0, we get

$$b_1 = \frac{a_1}{c_1} \quad \text{and} \quad b_2 = \frac{a_2 - b_1 c_2}{c_1},$$

from which the iteration formulas are derived, namely,

$$b(1,j+1) = \frac{a_1}{c(1,j)}$$

$$b(2,j+1) = \frac{a_2 - b(1,j+1)c(2,j)}{c(1,j)}$$

For the first set of values, we may let $b(1,1)$ and $b(2,1)$ be

$\frac{a_1}{a_3}$ and $\frac{a_2}{a_3}$, respectively, unless $a_3 = 0$.

3. Luke and Ufford's Extension of (1) [9]

Instead of deriving the formulas that will enable us to get an estimate of one factor at each stage of iteration, this extension derives the formulas that will display the estimates of two or more, possibly all the factors, at each stage of the iteration. This, therefore, eliminates the contamination of one factor by another due to division in the depression of the polynomials and perhaps due to the inaccuracy of a factor obtained. In the said iteration, if some sets of coefficients turn out to be convergent, while others don't, then depression of polynomials will be inevitable, so the rest of the factors can be obtained by similar procedure from the depressed polynomial.

However, to attempt to write a general expression or set of formulas to separate a polynomial into any number of factors presents considerable algebraic difficulties. So the iterative formulas are set up according to the individual problem.

To illustrate for a septic $\sum_1^7 a_i x^{i-1}$ seeking three quadratic factors, let the quadratics be $x^2 + b_2x + b_1$, $x^2 + d_2x + d_1$, and $x^2 + e_2x + e_1$. Dividing $f(x)$ by $x^2 + b_2x + b_1$, we obtain as in Lin's Method a set of formulas for the quotient coefficients. If the quotient is $\sum_1^5 c_i x^{i-1}$ with remainder $r_x x + r_1$, the recursive relations are:

$$c_4 = a_6 - b_2$$

$$c_3 = a_5 - c_4 b_2 - b_1$$

$$c_2 = a_4 - c_3 b_2 - c_4 b_1$$

$$c_1 = a_3 - c_2 b_2 - c_3 b_1$$

$$\hline r_2 = a_2 - c_1 b_2 - c_2 b_1$$

$$r_1 = a_1 - c_1 b_1$$

Dividing further $\sum_1^5 c_i x^{i-1}$ by $x^2 + d_2x + d_1$, we derive a similar set of formulas for the coefficients of the quotients $x^2 + e_2x + e_1$:

$$e_2 = e_4 - d_2$$

$$e_1 = c_3 - e_2 d_2 - d_1$$

$$\hline \bar{r}_2 = c_2 - e_1 d_2 - d_1$$

$$\bar{r}_1 = c_1 - e_1 d_1$$

To obtain a set of formulas for b's, d's and e's from which the iterative formulas are to be based, the idea is to set the remainder

coefficients r 's and $\bar{r}$'s to 0 and eliminate the c 's using the above equations. Thus by first setting $\bar{r}_1$ and $\bar{r}_2$ to 0, we get

$$d_1 = \frac{c_1}{e_1} \quad \text{and} \quad d_2 = \frac{c_2 - e_2 d_1}{e_1}.$$

to eliminate c_1 and c_2 , use the first group of formulas above to obtain

$$d_1 = \frac{a_3 - a_4 b_2 + a_5 b_2^2 + 2a_6 b_1 b_2 - 3b_1 b_2^2 - a_5 b_1 + b_1^2 - a_6 b_2^3 + b_2^4}{e_1}$$

$$d_2 = \frac{a_4 - a_5 b_2 - a_6 b_2^2 - b_2^3 + 2b_1 b_2 - a_6 b_1 - e_2 d_1}{e_1}.$$

Similarly, by setting r_1 and r_2 to 0, and eliminating the c 's,

$$b_1 = \frac{a_1}{d_1 e_1}, \quad b_2 = \frac{a_2 - b_1 e_1 d_2 - b_1 e_2 d_1}{d_1 e_1} \quad \text{and}$$

$$e_2 = a_6 - b_2 - d_2, \quad e_1 = a_5 - b_1 - b_2 a_6 + b_2^2 - d_1 - d_2 e_2.$$

So the iterative formulas can now be set up for the b 's, d 's and e 's in terms of their previous values and of the original coefficients.

For the case of linear factors, $(x+x_i)$, $i = 1, \dots, n$, the formulas are

$$x_n = a_n - \Sigma x_i$$

$$x_{n-1} = [a_{n-1} - a_n \Sigma x_i + \Sigma(x_i^2 + x_i x_j)]/x_n$$

$$x_{n-2} = [a_{n-2} - a_{n-1} \Sigma x_i + a_n \Sigma(x_i^2 + x_i x_j) - \Sigma(x_i^3 + x_i^2 x_j)]/x_{n-1} x_n$$

$$x_{n-3} = [a_{n-3} - a_{n-2} \Sigma x_i + a_{n-1} \Sigma (x_i^2 + x_i x_j) - a_n \Sigma (x_i^3 + x_i^2 x_j) + \Sigma (x_i^4 + x_i^3 x_j)] / x_{n-2} x_{n-1} x_n$$

.....

$$x_1 = a_1 / x_2 x_3 \dots x_n$$

where the notation Σ is used in the symmetric function sense, it being understood that in the formula for x_p ,

$$\Sigma (x_i^k + x_i^{k-1} x_j) = \sum_{\substack{i,j=1 \\ i \neq j}}^p (x_i^k + x_i^{k-1} x_j).$$

4. Bairstow's Method for Determining a Quadratic Factor [6]

From the recursive relations among the coefficients of the

quotient $\sum_1^{n-1} b_i x^{i-1}$ of $f(x)$ by the quadratic factor $x^2 + px + q$, namely,

$b_{n-k-1} = a_{n-k+1} - pb_{n-k} - qb_{n-k+1}$, $k = 1, \dots, n-2$, $b_{n-1} = 1$, $b_n = 0$, we get the remainder terms with coefficients

$$r = a_2 - pb_1 - qb_2 = b_0$$

$$s = a_1 - qb_1 = b_{-1} + pb_0$$

where b_0 and b_{-1} are obtained by taking $k = n-1$, n , respectively.

We require r and s , which are actually functions of p and q , say $f_1(p, q)$ and $f_2(p, q)$, respectively, to go to 0. To do this, express the functions f_1 and f_2 as Taylor Series in two variables, neglecting all terms beyond the first three, and equate to 0:

$$0 = f_1(p, q) \approx f_1(p_k, q_k) + \Delta p f_{1p}(p_k, q_k) + \Delta q f_{1q}(p_k, q_k) \quad \text{and}$$

$$0 = f_2(p, q) \approx f_2(p_k, q_k) + \Delta p f_{2p}(p_k, q_k) + \Delta q f_{2q}(p_k, q_k) \quad \text{where}$$

(p_k, q_k) is the estimate for (p, q) at iteration k , $\Delta p = p_{k+1} - p_k$,
 $\Delta q = q_{k+1} - q_k$, $f_{ij}(p_k, q_k)$ denotes $\frac{\delta f_i}{\delta j}$ evaluated at (p_k, q_k) .

Δp and Δq may be solved for simultaneously from the above two equations as follows:

Substituting the values for the partials of f_1 and f_2 with respect to p and q , as obtained from the equations for r and s ,

$$0 = b_0 + \Delta p \frac{\delta b_0}{\delta p} + \Delta q \frac{\delta b_0}{\delta q}$$

$$0 = b_{-1} + p b_0 + \Delta p \left(\frac{\delta b_{-1}}{\delta p} \right) + p \frac{\delta b_0}{\delta p} + b_0 + \Delta q \left(\frac{\delta b_{-1}}{\delta q} \right) + p \frac{\delta b_0}{\delta q}$$

Adding $(-p)$ times the first equation to the second, we obtain

$$(1) \quad 0 = b_0 + \Delta p \frac{\delta b_0}{\delta p} + \Delta q \frac{\delta b_{-1}}{\delta q}$$

$$(2) \quad 0 = b_{-1} + \Delta p \left(\frac{\delta b_{-1}}{\delta p} + b_0 \right) + \Delta q \frac{\delta b_{-1}}{\delta q}.$$

Now to obtain the partials of b_0 and b_{-1} , we may use the recursive relations

$$\frac{\delta b_{n-k-1}}{\delta p} = -p \frac{\delta b_{n-k}}{\delta p} - q \frac{\delta b_{n-k+1}}{\delta q} - b_{n-k} \quad \text{and}$$

$$\frac{\delta b_{n-k-1}}{\delta q} = -p \frac{\delta b_{n-k}}{\delta q} - q \frac{\delta b_{n-k+1}}{\delta q} - b_{n-k+1} \quad \text{which}$$

are obtained from the recursive formulas of the b 's above. However, for computational purposes, it is worthy of note that if the quotient

$\sum_{i=1}^{n-1} b_i x^{i-1}$ is further divided by $x^2 + px + q$, we get as before the

recursive relation $c_{n-k-3} = b_{n-k-1} - pc_{n-k-2} - qc_{n-k-1}$ for the coefficients of the quotient. Replacing k by $k-1$, and comparing with the first recursive relation for the partial of b 's (that is, with respect to p), we see that

$$\frac{\delta b_{n-k-1}}{\delta p} = -c_{n-k-2};$$

similarly, replacing k by $k-2$, and comparing with the second (that is, partials with respect to q),

$$\frac{\delta b_{n-k-1}}{\delta q} = -c_{n-k-1}.$$

Thus (1) and (2) above reduce to

$$(1) \quad b_{-1} + \Delta p(-c_{-2} + b_0) + \Delta q(-c_{-1}) = 0$$

$$(2) \quad b_0 + \Delta p(-c_{-1}) + \Delta q(-c_0) = 0$$

from which we derive the formulas

$$\Delta p = \frac{b_0 c_{-1} - c_0 b_{-1}}{c_{-1}^2 + c_0(b_0 - c_{-2})} \quad \text{and}$$

$$\Delta q = \frac{b_{-1} c_{-1} + b_0(b_0 - c_{-2})}{c_{-1}^2 + c_0(b_0 - c_{-2})}.$$

If $|\Delta p| + |\Delta q| < \epsilon$, a small preassigned number, then the iteration has converged to the desired p and q ; otherwise, let $p_{k+1} = \Delta p + q_k$ and $q_{k+1} = \Delta q + q_k$, and repeat the procedure.

To recapitulate the procedure:

(a) Take a starting value for p as well as for q .

(b) Obtain $b_{n-2}, \dots, b_0, b_{-1}$ using

$$b_{n-k-1} = a_{n-k+1} - pb_{n-k} - qb_{n-k+1},$$

$k = 1, \dots, n$ where $b_n = 0$ and $b_{n-1} = 1$.

(c) Derive $c_{n-4}, \dots, c_0, c_{-1}, c_{-2}$ from the relation

$$c_{n-k-3} = b_{n-k-1} - pc_{n-k-2} - qc_{n-k-1},$$

$k = 1, \dots, n-1$.

(d) Use c_0, c_{-1}, c_{-2} to obtain Δp and Δq from

$$\Delta p = \frac{b_0 c_{-1} - c_0 b_{-1}}{c_{-1}^2 + c_0(b_0 - c_{-2})} \quad \text{and}$$

$$\Delta q = \frac{b_{-1} c_{-1} + b_0(b_0 - c_{-2})}{c_{-1}^2 + c_0(b_0 - c_{-2})}.$$

(e) Test for $|\Delta p| + |\Delta q|$:

If $< \epsilon$, take (p_k, q_k) to be the desired (p, q) and solve the equation $x^2 + px + q$ for the zeros. Otherwise, repeat the iteration using $p_{k+1} = \Delta p + p_k$ and $q_{k+1} = \Delta q + q_k$, unless the number of iterations has exceeded the upper limit set at the start.

5. Graeffe's Root Squaring Method for $f(x)$ Having Complex Zeros [12]

From $f(x) = \sum_{i=1}^{n+1} a_i x^{i-1}$, derive $g(x) = f(\sqrt{x})$ whose zeros

are squares of the zeros of $f(x)$. For instance, for $n = 4$,

$x^4 + a_4 x^3 + \dots + a_1$ is transformed into $x^4 + (-a_4^2 + 2a_3)x^3 +$

$(2a_1 - 2a_4a_2 + a_3^2)x^2 + (2a_1a_3 - a_2^2)x + a_1^2$ so that if

$g(x) = \sum_{i=1}^{n+1} b_i x^{i-1}$, we have the formulas

$$b_4 = -a_4^2 + 2a_3$$

$$b_3 = a_3^2 - 2a_4a_2 + 2a_1$$

$$b_2 = -a_2^2 + 2a_3a_1$$

$$b_1 = a_1^2.$$

If the zeros of $f(x)$ differ in absolute value, the zeros of $g(x)$ must differ by a greater amount. Thus by doing this transformation m times, an equation may be found with roots equal to the $2m$ th power of the original and which roots differ in absolute value as greatly as may be desired. In this case, the coefficient of x^{n-2} will nearly be the square of the absolute value of the zeros with largest magnitude, since if $a + bi$ are those zeros, the $\sum r_i r_j$ is dominated by $a^2 + b^2$. The greater m is, the more nearly true this is. The root-squaring process is terminated when the coefficient of x^{n-2} in the last derived equation is nearly equal that in the previous step.

Having found in this manner the $4m$ th power of the largest absolute value among the zeros of the original, the square of such absolute value may be obtained by logarithm. The zeros can then be determined from this by using relation among the zeros.

To illustrate for $n = 4$, if the zeros are $c_j \pm d_j i$, $j = 1, 2$, then $a_1 = (c_1^2 + d_1^2)(c_2^2 + d_2^2)$ so that since $|r_1|^2 = c_1^2 + d_1^2$ is known, the square of the absolute values of the second pair of zeros, $|r_2|^2 = c_2^2 + d_2^2$ can be determined. Then using the relation $a_4 = -2(c_1 + c_2)$ and $a_2 = -2(|r_1|^2 c_2 + |r_2|^2 c_1)$, c_1 and c_2 can be solved for simultaneously, and consequently from the values of $|r_1|^2$ and $|r_2|^2$, d_1 and d_2 can be determined.

6. Hitchcock's GCD Method for $f(x)$
Having All Zeros Complex [4,5]

First assume the coefficients a_i of $f(x)$ real.

By dividing $f(x) = \sum_1^{n+1} a_i x^{i-1}$ by $x^2 + q$, we obtain the identity

$$f(x) = [x^{n-2} + a_n x^{n-3} + (a_{n-1} - q)x^{n-4} + \dots][x^2 + q] +$$

$$[a_2 - a_4 q + a_6 q^2 - \dots \pm a_n q^{m-1}]x + [a_1 - a_3 q + a_5 q^2 - \dots \mp q^m]$$

where $m = n/2$ and the $\pm$ in the last term of each of the last two coefficient expressions is determined such that the signs are alternating.

If $\pm \sqrt{q}i$ were a pair of zeros of $f(x)$, then the last two coefficients expressions would be identically 0. Hence we derive the two test equations for a pair of pure imaginary zeros, namely,

$$(1) \quad a_2 - a_4 q + a_6 q^2 - \dots \pm a_n q^{m-1} = 0$$

$$(2) \quad a_1 - a_3 q + a_5 q^2 - \dots \mp q^m = 0$$

So given $f(x)$, if the above polynomials (1) and (2) have a common factor, say $q-q_1$, then we shall have determined a pair of zeros of $f(x)$, namely, $\pm \sqrt{q_1} i$. The GCD algorithm can be employed for this purpose. However, in most cases the common factor is 1. So the idea now is to use Horner's root diminishing process and move a pair of zeros of $f(x)$ until they are transformed to pure imaginary ones.

The process then boils down to using the GCD algorithm on the test equations. If the last real number remainder in the algorithm is positive (or negative), try translation of variable by a ~~negative~~ (or ~~positive~~) number h , and apply the GCD algorithm again on the new test equations. If the remainder changes sign, then we know there is a pair of zeros whose real part lies between 0 and $+h$. So try a number between 0 and h for a translating factor, and repeat the process. Continuing this way, we can thus narrow down the interval containing the desired real part by translations till the remainder in the GCD algorithm approximates 0. This way, we also find at the same time the imaginary part of that particular pair of zeros from the common factor of the two test polynomials. We may also employ interpolation to estimate the amount of translation necessary, after we have tried a couple of values for h . Thus if translation of variable by h gives a common factor $x-q$ to the test polynomials, the desired zeros are then $+h \pm \sqrt{q} i$.

For a_1 's complex, the procedure is practically the same except that there is a different set of test equations for a pure imaginary zero q_i . If $f(x) = x^n + (a_n + b_1)x^{n-1} + \dots (a_1 + b_1 i)$, then the

remainder in dividing $f(x)$ by $x - q_i$ is $f(q_i)$, which is what we need to test whether 0 or not. By separating real and imaginary parts of $f(q_i)$ the two test equations then are

$$(1) \quad a_1 - b_2q - a_3q^2 + b_4q^3 + \cancel{a_5}q^4 - b_6q^5 - a_7q^6 \dots = 0$$

$$(2) \quad b_1 + a_2q - b_3q^2 - a_4q^3 + b_5q^4 + a_6q^5 - b_7q^6 - \dots = 0$$

where it will be noted that the signs are 2 plus and 2 minus alternately except at the start of (1).

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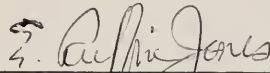
BIOGRAPHICAL SKETCH

Alicia Hidalgo Santos was born on May 20, 1937, at Biñan, Laguna, Philippines. In April, 1948, she graduated as class valedictorian from Biñan Elementary School, and four years hence, from Lake Shore High School with the same honors. She received the degree of Bachelor of Science in Education from the University of Philippines in 1956 (^{#45}magna cum laude). From 1956 to 1957, she taught as the U. P. College of Education, after which she transferred to the U. P. College of Liberal Arts, with which she has been connected up to the present time. She was granted a U.P. local fellowship from 1959 to 1961, during which time she obtained a Master of Science degree, major in Mathematics. In September, 1963, she became a Philippine National Science Development Board fellow under Project 2.28.1, which enabled her to pursue her work toward the degree of Doctor of Philosophy.

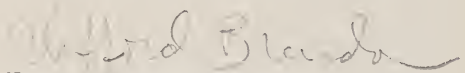
Alicia Hidalgo Santos is married to Carlos Soriano Santos, an architect, and is the mother of two children. She is a member of The Mathematical Association of America, the Phi Kappa Phi International Honor Society, and the Pi Gamma Mu International Honor Society.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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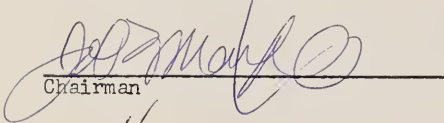


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